CAPITAL UNIVERSITY OF SCIENCE AND TECHNOLOGY ISLAMABAD

Fuzzy Mappings in b-Metric Spaces

by

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Declaration of Authorship

I, Umarah sadaf, declare that this thesis titled, 'Fuzzy Mappings in b-Metric Spaces' and the work presented in it are my own. I confirm that:

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- Where I have consulted the published work of others, this is always clearly attributed.

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_____

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_____

بِسْمِ اللهِ الرَّحْمَنِ الرَّحِيمِ

In the Name of Allāh, the Most Gracious, the Most Merciful

Al-FATIHA

All praises be to ALLAH, the Sustainer of all the worlds.

Most Gracious, most Merciful.

Thee do we worship and Thine aid we seek.

Show us the straight way.

The way of whom Thou hast bestowed Thy Grace, not of those who earned Thy wrath and who went astray.

"By and large it is uniformly true that is mathematics there is a time lapse between a mathematical discovery and the moment it becomes useful; and that this lapse can be anything from 30 to 100 years, in some cases even more; and that the whole system seems to function without any direction, without any reference to usefulness and without any desire to do things which are useful."

Jhon Von Neumann (1903-1957)

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Abstract

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A number of authors proved common fixed point theorems for fuzzy mappings in metric spaces under certain contraction conditions. In high energy physics, these theorems are helpful to resolve geometric problems. In the present work, we extend some common fixed point theorems for fuzzy mappings on metric spaces to common fixed point for fuzzy mappings on b-metric spaces. Our results will helpful in solving certain fixed point problems in b-metric spaces.

Acknowledgements

In the name of Allah, the Most Gracious and Most Merciful

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Symbols

(X, d)	Metric space	
(X, d_b)	b-Metric space	
$(0, 1)$	Open interval	$(0, 1) = \{x/0 < x < 1\}$
$[0, 1]$	Close interval	$[0, 1] = \{x/0 \leq x \leq 1\}$
$\square$	QED	Which was to be proven
Σ	Sigma	Summation
Π	pi	
α	Alpha	
β	Beta	
$\lim_{x \to \infty}$	Limit	
$\Rightarrow$	Implies that	
$\forall$	For all	
$\exists$	There exist	
∞	Infinity	
ϵ	Epsilon	
η	Eta	
δ	Delta	
ξ	Xi	
μ	Mu	
ν	Nu	
λ	Lambda	
$\in$	Belong to	
ϕ	Fi	

ψ	Si
$\hat{M}$	Induced by M
$\hat{N}$	Induced by N
$\mathcal{H}$	Hausdorf metric
$\mathbb{R}$	Set of real numbers
$\mathbb{N}$	Set of natural numbers
$\mathbb{Q}$	Set of rational numbers
$\mathbb{C}$	Set of complex numbers
$\nRightarrow$	Does not implies that
$\Leftrightarrow$	If and only if
$\longrightarrow$	Approaches to
$<$	Less then
$>$	Greater then
$\leq$	Less or equal to
$\geq$	Greater or equal to
$\neq$	Not equal

Dedicated

To

`My Mother`

A Strong and gentle soul who taught me to trust in
Allah, believe in hard work and that so much could be
done with little

`My Grandmother(late)`

For being my first Teacher

`My Father`
For earning a honest living for us and for supporting
and encouraging me to believe in myself

Chapter 1

Introduction

In mathematics, the existence of solution is same as the existence of fixed point of a corresponding map. Fixed point theory gives suitable conditions for the existence of solution of a problem. Therefore fixed point has broad importance in certain fields of mathematics and other sciences, for example many problems in different fields of sciences can be transformed into the " problem of fixed point". The theory is also an attractive mixture of pure and applied mathematics including topology and geometry. In the study of non-linear analysis fixed point theory is considered as a fundamental tool.

In the beginning, it was considered that the fixed point theory is pure analytical theory, later on the theory can be classified into different fields that are metric, discrete and topological fixed point theory. Banach fixed point theorem [8] or Banach contraction principle is supposed to be the most valuable and adaptable consequence in metric fixed point theory. Banach fixed point theorem stats as,

" On a complete metric space, a contraction mapping has a unique fixed point. More precisely, (X, d) is a complete metric space and T is a self map on X such that,

$$d(Tx, Ty) \leq \alpha d(x, y), \quad \forall x, y \in X, \alpha \in [0, 1)$$

Then T has a fixed point".

This theorem is the most important result in mathematical analysis. A number of researchers in mathematics [11], [14], [17], [37], [43] are fascinated to Banach contraction principle by virtue of its simplification and generalization.

Zadeh[55] esteblished the idea of fuzzy sets in 1965. Fuzzy control seems to be the most useful tool from application point of view. For the applications, where the exact quantitative representation of some particular samples is inappropriate or impossible, the implementation of fuzzy theory seems to be an easy and convinent. Hence, for real time performance in many models designes we often find the use of fuzzy methods. There are many applications of fuzzy objects and methods including regulation, production control, household applience and music [28]. Fuzzy logics and approximate reasoning related to fuzzy logic are the theoretical and methodoligical background of fuzzy mathematics.

In this work, we deals with the aspects of fuzzy mappings. The fuzzy mappings are considered as functions, that alocate a fuzzy set to an element defined on a given domain. Then introduction of the notion of fuzzy mappings is given by the Heilpern [22], he also proved the fixed point theorem for fuzzy mappings in the settings of metric spaces that is the extended form of the Banach contraction principle. Later on, a variety of papers are seen on fixed point theorems for fuzzy mappings that satisfied the contractive conditions, some of these papers are [32], [34], [41]. For the the applications of integral equations and partial differential equations, a number of Mathematicians and Engineers focused on fixed point results for fuzzy mappings. Fuzzy control seems to be the most useful from application point of view.

The notion of b-metric space was first appeared in the work of Bakhtin [9], then used by Czerwik [47]. M.Boriceanu [30] also found examples and some fixed point theorems in b-metric spaces. Afterward, variational principle in b-metric spaces is formulated by Ekeland [31] in 1974. In the proof of fixed point theorem in complete metric space, Ekeland's variational principle is used as a main tool.

Recently, many authors developed fixed point theory in b-metric spaces. Some of these authors [16], [29] and [52] focused on the topological properties of b- metric

spaces, which proved the concept that every b- metric space defined on a topology which is induced by the convergent of b-metric space is the semi-metrizable space. Therefore, the use of different aspects of b- metric space in literature is obvious. Note that, 'a b- metric space is always considered to be a topological space in the sense of topology which is induced by the convergent of b- metric space'.

In this thesis, several papers are reviewed which are mentioned above but our main focus is on paper titled as "Common fixed point theorems for fuzzy mappings in metric space" by T. Kamran [23]. This paper is the corrected form of paper [1]. After detailed study of litrature related to this paper [23] we extended the results in the settings of b- metric spaces.

Rest of the thesis is detailed as follows:

• **Chapter** 2; includes basic tools about metric spaces, b- metric spaces, fixed point and fuzzy mappings.

• **Chapter** 3; is about literature review and detailed study of common fixed point theorems for fuzzy mappings in metric spaces.

• **Chapter** 4; is the extension of the results of Chapter 3 from metric spaces to b-metric spaces.

Chapter 2

Preliminaries

In this chapter,we discuss some basic definitions and concept which we have to use in this thesis. **Section** 2.1 covers some basics of metric space and examples of various concepts. **Section** 2.2 concerns with the study related to b-Metric spaces and examples of b-metric space. This section also includes Cauchy sequences and completeness criteria of b-metric spaces. **Section** 2.3 of this chapter deals with the fixed points in metric space. **Section** 2.4 concerns with some types of mappings. In the last **Section**, we focus on fuzzy mappings.

2.1 Metric Spaces

This section concerns with the definition and examples of metric space, bounded and unbounded sets, supremum, infimum, maximum and minimum of sets, Hausdorff metric, sequences in metric space, Hausdorff metric, Cauchy sequence and completeness of metric space.

Definition 2.1.1. (Metric Space)

A function $d : X \times X \to \mathbb{R}_+$ (where X denotes a non-empty set and the set of non-negative real numbers is denoted by $\mathbb{R}_+$) is called metric (or distance function), if it satisfies the following properties.

(M1): For any pair $p_1, p_2 \in X$, $d(p_1, p_2) \geq 0$ and $d(p_1, p_2) = 0 \Leftrightarrow p_1 = p_2$

(M2): For any pair $p_1, p_2 \in X$, $d(p_1, p_2) = d(p_2, p_1)$ (symmetric property)

(M3): For $p_1, p_2, p_3 \in X$, $d(p_1, p_2) \leq d(p_1, p_2) + d(p_2, p_3)$ (Triangle inequality).

The pair (X, d) is then called a metric space on X.

Example 2.1.2. 1. (Real line $\mathbb{R}$) The set of all real numbers is denoted by $\mathbb{R}$. Define a metric $d : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ as follows,

$$d(r_1, r_2) = \mid r_1 - r_2 \mid; \quad \forall \quad r_1, r_2 \in \mathbb{R}.$$

Then the pair $(\mathbb{R}, d)$ is a metric space and d is called the standard or usual metric on $\mathbb{R}$.

2. (Euclidean space $\mathbb{R}^n$) This space is also called n-dimensional Euclidean space and can be obtained by taking all ordered n-tuples of real numbers, *i.e*,

$$x = (\xi_1, \xi_2, \cdots, \xi_n), \quad y = (\eta_1, \eta_2, \cdots, \eta_n).$$

$(\mathbb{R}^n, d)$ is a metric space with Euclidean matric d defined as;

$$d(x, y) = \sqrt{(\xi_1 - \eta_1)^2 + \cdots + (\xi_n - \eta_n)^2}.$$

3. (Function space $C[a_1, a_2]$) The space of all real valued continuous functions is denoted by $C[a_1, a_2]$

(where $I = [a_1, a_2]$ is a closed interval of real numbers).

Define a metric $d : C[a_1, a_2] \times \mathbb{C}[a_1, a_2] \to \mathbb{R}$ by,

$$d(p, q) = \max_{t \in I} \mid p(t) - q(t) \mid .$$

Then $(C[a_1, a_2], d)$ is a metric space and d is a metric on $C[a_1, a_2]$.

4. (ℓ^2 space) ℓ^2 space is also called Hilbert sequence space, the element of ℓ^2 is a sequence $x = (\xi_i), i = 1, 2, \therefore, \infty$ of numbers such that,

$$\sum_{i=1}^{\infty} |\xi_i|^2 < \infty.$$

Define $d : \ell^2 \times \ell^2 \to \mathbb{R}$ by,

$$d(x, y) = \left(\sum_{i=1}^{\infty} |\xi_i - \eta_i|^2 \right)^{1/2}.$$

Then the pair (ℓ^2, d) is a metric space.

Definition 2.1.3. (Bounded and Unbounded Sets)

Let A be a subset of real numbers $\mathbb{R}$, then A is bounded if there exist a real number M such that,

$$a \leq M; \text{for all} \quad a \in A.$$

The real number M is called upper bound of A. if there exist no such M, then A is said to be unbounded from above. Let B be the set of all upper bounds of A having the smallest number M. Then the number M is known to be the least upper bound (l.u.b) or supremum of set A.

Furthermore, a set may or may not have a Supremum and if the supremum of a set exists it will be unique.

The supremum M of a set A has the following properties.

(i) M must be the upper bound of A, *i.e*, $a \leq M \quad \forall a \in A$.

(ii) For a small positive real number there exists a number $a \in A$ such that,

$$a > M - \epsilon$$

A subset A of numbers $\mathbb{R}$ is called bounded bellow, if there exist real number m such that,

$$m \leq a; \text{for all} \quad a \in A.$$

The real number m is called lower bound of A. In case if there exists no such m then set A is called unbounded from below. Let B be the set of all lower bounds of A having the smallest number m.

Then the number m is known as greatest lower bound or infimum of set A.

Moreover, a set may or may not have a infimum, if the infimum of a set exists it will be unique. The infimum m of a set A has following properties:-

(i) m must be the lower bound of A, *i.e*,

$$m \leq a, \quad \forall \quad a \in A$$

(ii) For a small positive real number ϵ, there exist a number $a \in A$ such that,

$$a < m + \epsilon$$

A bounded set is bounded above as well as bounded below for instance see the following examples:

1- The set of natural numbers is denoted by $\mathbb{N}$ is bounded below but unbounded above, therefore it is not bounded and it has no supremum.

2. A set which contains finite numbers is bounded.

3. Infinite set $A = \{a : 0 \leq a \leq 2 \forall a \in \mathbb{Q}\}$ is bounded.

4. The set of real numbers is denoted by $\mathbb{R}$; is not bounded.

Definition 2.1.4. (Partially Ordered Set)

A non-empty set U is called a Partially ordered set, if it satisfies following properties with binary operation $\preceq$,

∗ Reflexive; For each $\lambda \in U \Rightarrow \lambda \preceq \lambda$,

∗ Antisymmetric; If $\lambda \preceq \mu$ and $\mu \preceq \lambda \Rightarrow \lambda = \mu, \quad \forall \lambda, \mu \in U$,

∗ Transitive; If $\lambda \preceq \mu$ and $\mu \preceq \nu \Rightarrow \lambda = \nu, \quad \forall \lambda, \mu, \nu \in U$.

For example power set $P(U)$ of a non-empty set U is partially ordered set. If each member of set U is comparable then U is called chain or totally ordered set.

Definition 2.1.5. (Maximum and Minimum)

Let $B = \beta_1, \beta_2, \beta_3...\beta_n$ be any set. If set B is in totally ordered set then largest and smallest values of B are called maximum and minimum of B respectively, denoted by $\max(B)$ or $\max_i(\beta_i)$ and $\min(B)$ or $\min_i(\beta_i)$.

Definition 2.1.6. (Sequence)

A sequence is a set of elements of any nature that are ordered as are the natural numbers $1, 2, 3, \cdots, n$, it can be written as $x_1, x_2, \cdots, x_n$ or simply $\{x_n\}$.

Definition 2.1.7. (Convergent Sequence)

Let (X, d) be a metric space, a sequence $\{x_n\}$ is said to be convergent, if there exist an $x \in X$ such that,

$$\lim_{n \to \infty} d(x_n, x) = 0,$$

then x in known as limit of $\{x_n\}$ and written as,

$$\lim_{n \to \infty} x_n = x \quad or \quad x_n \to x.$$

Definition 2.1.8. (Cauchy Sequence)

Let (X, d) be a metric space, a sequence $\{x_n\}$ is said to be if there exist a natural number $\mathbb{N}$ for every $\epsilon > 0$ such that,

$$d(x_m, x_n) < \epsilon \quad \text{for each} \quad n, m > \mathbb{N}.$$

Definition 2.1.9. (Complete Space)

A metric space X is said to be complete, if every Cauchy sequence in metric space

X converges to a point in X.

For example, the real line $\mathbb{R}$ and the complex plan $\mathbb{C}$ are complete metric spaces.

Definition 2.1.10. (The Hausdorff distance)

Let A, B be the non-empty compact bounded subsets of X in a metric space (X, d) Then the Hausdorff distance between A and B is given as,

$$\mathcal{H}(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B}(b, A)\}$$

2.2 b-Metric Space

In 1989, Bakhtin [9] introduced the notion of b-metric space. Then several authors did work on fixed point theory in b-metric spaces. In this section, we concern with the definition, examples and other aspects related to b-metric spaces.

Definition 2.2.1. (b-Metric Space)

Let X be a non-empty set and let $b \geq 1$ be a given real number, a function $d_b : X \times X \mapsto \mathbb{R}$ is called a b-metric, if it satisfies following properties;

(B1): For any pair $r_1, r_2 \in X$, $d_b(r_1, r_2) = 0 \Longleftrightarrow r_1 = r_2$.

(B2): For any pair $r_1, r_2 \in X$, $d_b(r_1, r_2) = d_b(r_2, r_1)$.

(B3): For $r_1, r_2, r_3 \in X$, $d_b(r_1, r_3) \leq b[d_b(r_1, r_2) + d_b(r_2, r_3)]$.

then the pair (X, d_b) is called a b-metric space. Where d_b is a b-metric on X.

Remark 2.2.2. If $b = 1$, then b-metric space will be a metric space.

Example 2.2.3. The set of real numbers $\mathbb{R}$ is a b-metric space with the metric defined as,

$$d_b(u, v) = \mid u - v \mid^2 \quad \forall u, v \in \mathbb{R}$$

$$\text{where} \quad b = 2.$$

Example 2.2.4. Let $\ell_p(\mathbb{R})$ be a set, where $0 < p < 1$ and

$$\ell_p(\mathbb{R}) = \{r_k\} \subseteq \mathbb{R}; \sum_{k=1}^{\infty} \mid r_k \mid^p < \infty$$

with the metric $d_b : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ defined as,,

$$d_b : \ell_p(\mathbb{R}) \times \ell_p(\mathbb{R}) \mapsto \mathbb{R}$$

defined as,

$$d_b(r,s) = (\sum_{k=1}^{\infty} \mid r_k - s_k \mid^p)^{\frac{1}{p}}$$

then $\ell_p(\mathbb{R})$ is a b-metric space with $b = 2^{\frac{1}{p}} > 1$.

Example 2.2.5. The $L^p[0,1]$ where p lies between 0 and 1 for all Real Valued Functions, $u(z), z \in [0,1]$ such that,

$$\int_0^1 \mid u(z) \mid^p < \infty$$

along with the metric $d_b : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ defined as,

$$d(u,v) = \left[\int_0^1 \mid u(z) - v(z) \mid^p dz \right]^{1/p}$$

$$\text{for all} \quad u, v \in L^P[0,1],$$

then $L^P[0,1]$ is a b-metric space with $b = 2^{1/p}$.

Example 2.2.6. Let $X_b = \{0, 1, 2\}$ with the metric $d_b : X_b \times X_b \mapsto \mathbb{R}$ defined as,

$$d_b(0,0) = d_b(1,1) = d_b(2,2) = 0$$

$$d_b(0,1) = d_b(1,0) = d_b(1,2) = d_b(2,1) = 1$$

$$\text{Also} \quad d_b(2,0) = d_b(0,2) = t$$

where $t \in \mathbb{R}$ and $t \geq 2$, therefore,

$$d_b(u, v) \leq \frac{t}{2}[d_b(u, w) + d_b(w, u)] \quad \forall u, v, w \in X_b$$

hence d_b is a b-metric on X_b with $b = \frac{t}{2}$. But for $t > 2$ triangular inequality cannot fulfilled.

Definition 2.2.7. (Convergent sequence in b- metric space)

The sequence $\{t_n\}_{n\in\mathbb{N}} \in X$ in a b- metric space (X, d_b) is called Convergent iff $t \in X, \forall \delta > 0, \exists$ an $n(\delta)$ in $\mathbb{N}$ such that $\forall n \geq n(\delta)$,

$$d_b(t_n, t) < \delta \quad or \quad \lim_{n \longrightarrow \infty} t_n = t.$$

Definition 2.2.8. (Cauchy sequence in b- metric space)

Let (X, d_b) be a b- metric space, then the sequence $\{t_n\}_{n\in\mathbb{N}}$ in X is called Cauchy sequence iff $t \in X, \forall \delta > 0$ then there exist an $n(\delta)$ in $\mathbb{N}$ such that, for every $n, m \geq n(\delta)$ we have,

$$d_b(t_n, t_m) < \delta$$

if every Cauchy sequence is convergent then the b-metric space (X, d_b) is said to be complete.

Definition 2.2.9. (Closedness in b-metric space)

A set $A \subset X$ is closed in a b-metric space (X, d_b), iff for every convergent sequence in $A, \exists$ an element $t \in A$

Definition 2.2.10. (Compact Sets)

A set $A \subset X$ is said to be Compact set in a b-metric space (X, d_b), iff there exist a sub-sequence for each sequence of element of A, that converges to an element of A.

Remark 2.2.11. Following statements hold in b-metric space (X, d_b).

1. There is a unique limit of convergent sequence in b-metric space.

2. In *b*-metric space, convergent sequence must be Cauchy sequence.

3. A *b*-metric is not continuous in general as illustrated by following example,

   Example 2.2.12. [27]

   "Let $X = \mathbb{N} \cup \{\infty\}$ and let $d_b : X \times X \to \{0, +\infty\}$ is defined by

$$
d_b(m, n) =
\begin{cases}
0 & \text{if } m = n, \\
|\frac{1}{m} - \frac{1}{n}| & \text{if one of } m, n \text{ is even and the other is even or } \infty, \\
5 & \text{if one of } m, n \text{ is odd and the other is odd or } \infty, \\
2 & \text{otherwise.}
\end{cases}
$$

   It can be checked that for all $m, n, p \in X$ we have

$$
d_b(m, p) \leq \frac{5}{2}[d_b(m, n) + d_b(n, p)]
$$

   Thus (X, d_b) is a *b*- metric space with $b = \frac{5}{2}$. Let $x_n = 2n$ for each $n \in \mathbb{N}$, then

$$
d_b(2n, \infty) = \frac{1}{2n} \longrightarrow 0 \text{ as } n \longrightarrow \infty
$$

   that is, $x_n \longrightarrow \infty$, but $d_b(x_n, 1) = 2 \nrightarrow 5 = d_b(\infty, 1)$ as $n \longrightarrow \infty$."

2.3   Fixed Points in Metric space

In the current section, we will describe the Fixed point in metric space and its examples. Fixed point theorem is one of the most adoptable tools in Mathematical Analysis, which is used to solve several problems in different fields of mathematics. Several authors essentially, [12],[20],[25] interpret fixed point theory on complete metric space. Recently, fixed point theory extends fast in partially ordered metric space.

Definition 2.3.1. (Fixed point)
A fixed point of a function $F \mapsto X \times X$ is an $x_0 \in X$ which mapped onto itself

that is, $t \in X$ is said to be a fixed point of function $f : X \mapsto X$ if and only if,

$$f(t) = t$$

Following are examples of fixed point.

Example 2.3.2. Let $f; \mathbb{R} \mapsto \mathbb{R}$ be a function defined as,

$$f(t) = t^2 - 3t + 4$$

then f has fixed point $t = 2$ because $f(2) = 2$.

Example 2.3.3. Let $X = \mathbb{R}$, a self map T from X into X in a metric space (X, d) defined as,

$$Tz = 2z + 1; \quad \forall z \in X$$

then $z = -1 \in \mathbb{R}$ is the only fixed point of Tz.

Example 2.3.4. Let I be the identity map on metric space $X = \mathbb{R}$ with usual metric d, *i.e,*

$$I(z) = z, \quad \forall z \in X$$

then every point of X will be the fixed point of I.

Example 2.3.5. There is no fixed point of T, if $Tz = z + 1$, because there is no solution for $z + 1 = z$.

Example 2.3.6. Table showing list of functions with their fixed points taken out from mathworld.wolfram.com/fixed point [51].

Function	Fixed Point	Function	Fixed Point
cosecant	1.1141571408	inverse hyperbolic cosecant	0.9320200293
cosine	0.7390851332	inverse hyperbolic cosine	–
cotangent	0.8603335890	inverse hyperbolic cotangent	1.1996786402
hyperbolic cosecant	0.9320200293	inverse hyperbolic secant	0.7650099545
hyperbolic cosine	–	inverse hyperbolic sine	0
hyperbolic cotangent	1.1996786402	inverse hyperbolic tangent	0
hyperbolic secant	0.7650099545	inverse secant	–
hyperbolic sine	0	inverse sine	0
hyperbolic tangent	0	inverse tangent	0
inverse cosecant	1.1141571408	secant	4.9171859252
inverse cosine	0.7390851332	sine	0
inverse cotangent	0.8603335890	tangent	4.4934094579

Geometrically, the fixed point of a single valued function $y = f(x)$ lies where the graph of the function f intersects with the real line $y = x$.

Thus a function may or may not have a fixed point. Furthermore, the fixed point may not be unique. Figure given bellow shows graph of $\mathbb{R}$ function having three fixed points.
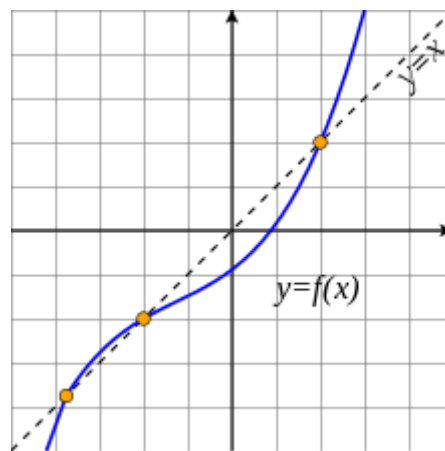

FIGURE 2.1: Three Fixed points

Example 2.3.7. Let $X = \mathbb{R}$ and T maps from X into X such that,

$$Tx = x + 1$$

then, T has no fixed point,
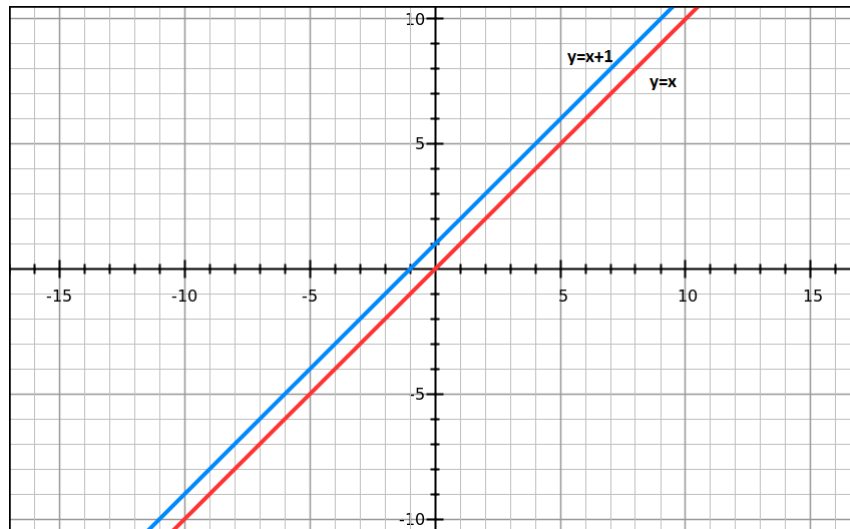
since $x + 1 = x$ has no solution.

FIGURE 2.2: No Fixed point

Example 2.3.8. Let $X = \mathbb{R}$ and T maps from X into X such that,

$$Tx = 2x + 1$$

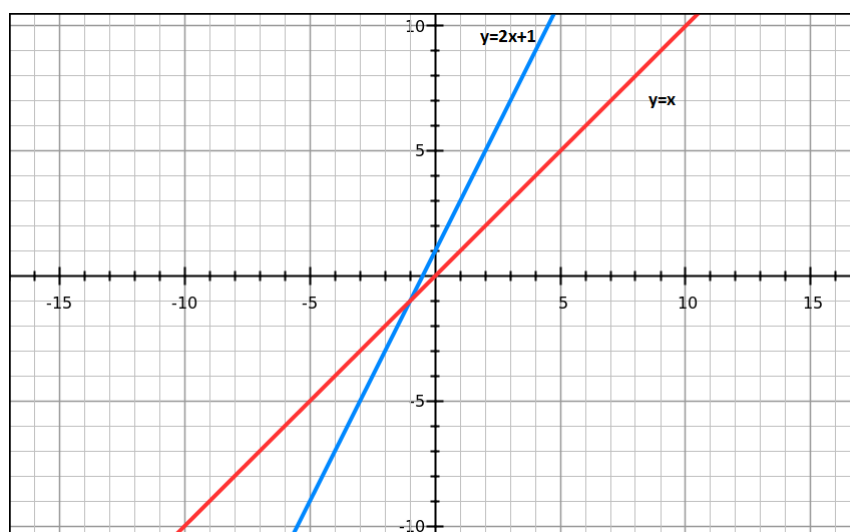then, T has a unique fixed point $x = -1$.

FIGURE 2.3: Unique Fixed point

Definition 2.3.9. (Zeroes of a Function)

Finding zeroes of a real valued function $g(x)$ defined on an interval is as finding the fixed point of $f(x)$ where,

$$f(x) = x - g(x)$$

since, zeroes of $g(x)$ means x such that,

$$g(x) = 0$$

$$\Rightarrow \quad x - g(x) = x$$

OR

$$f(x) = x, \quad i.e \quad \text{"}x \quad \text{is a fixed point of} \quad f(x)\text{"}$$

Example 2.3.10. Consider the quadratic polynomial,

$$g(x) = x^2 + 5x + 4$$

therefore, zeroes of $g(x)$ are

$$x = -4, \quad x = -1$$

Resulting, $g(x) = 0$ as,

$$x^2 + 5x + 4 = 0$$

$$x^2 + 4 = -5x$$

$$x = \frac{x^2 + 4}{-5} = f(x)$$

Clearly; problem of finding zeroes of $g(x)$ is equivalent to the problem of finding the fixed point of $f(x)$ such that $x = f(x)$.

Example 2.3.11. Indeed an operator equation $T_x = y$ may be equivalently transformed to fixed point problem, that is;

$$S(x) = x \quad with \quad S(x) = x + T_x - y$$

Therefore, finding a vector x such that $S(x) = x$ is same as finding or solving the equation,

$$x + T_x - y = x$$

$$\Rightarrow \quad T_x - y = 0$$

$$\Rightarrow \quad T_x = y$$

2.4 Types of Mappings

Definition 2.4.1. (Lipschitzian Mapping)
A self map $M : X \mapsto X$ on a metric space, then M is a Lipschitzian Map if $\exists \lambda \geq 1$ such that,

$$d(M(\eta_1), M(\eta_2)) \leq \lambda d(\eta_1, \eta_2); \quad \forall \eta_1, \eta_2 \in X$$

Here λ is called Lipschitzian constant.

Example 2.4.2. Let $M : X \mapsto X$ be a self map on $X = \mathbb{R}$ defined as $M(t) = 5t \quad \forall t \in X$, then

$$d(M(t_1), M(t_2)) = d(5t_1, 5t_2)$$
$$= \mid 5t_1 - 5t_2 \mid$$
$$= 5 \mid t_1 - t_2 \mid$$
$$= 5d(t_1, t_2)$$

Here $\lambda = 5$ is the Lipschitzian constant.

Definition 2.4.3. (Contraction Mapping)
A self map $M : X \mapsto X$ on a metric space, is a Contraction Mapping if its

Lipschitzian constant $\lambda < 1$ $i.e$ $0 \leq \lambda < 1$ such that,

$$d(M(\eta_1), M(\eta_2)) \leq \lambda d(\eta_1, \eta_2); \quad \forall \eta_1, \eta_2 \in X \quad \eta_1 \neq \eta_2$$

Example 2.4.4. A self map $M(X) = t^3$ $\forall t \in X$ on a metric space, $X = (0, \frac{1}{2})$ with metric d defined as $\sup(0, \frac{1}{2}) \Rightarrow \mid t \mid < \frac{1}{2}$ then for $\eta_1, \eta_2 \in X, d(\eta_1, \eta_2) = \mid \eta_1 - \eta_2 \mid$,

$$\begin{aligned}
d\left(M(\eta_1), M(\eta_2)\right) &= \mid M(\eta_1) - M(\eta_2) \mid \\
&= \mid \eta_1^3 - \eta_2^3 \mid \\
&= \mid \eta_1^2 + \eta_1\eta_2 + \eta_2^2 \mid\mid \eta_1 - \eta_2 \mid \\
&\leq \frac{1}{2} d(\eta_1, \eta_2) \\
&\leq \lambda d(\eta_1, \eta_2)
\end{aligned}$$

Since $\lambda \in (0, \frac{1}{2})$ therefore M is a contraction.

Definition 2.4.5. (Contrative Mapping)

A self map $M : X \mapsto X$ on a metric space is a Contrative Mapping if,

$$d(M(\eta_1), M(\eta_2)) < d(\eta_1, \eta_2); \quad \forall \eta_1, \eta_2 \in X \quad \eta_1 \neq \eta_2$$

Every contraction is contractive mapping but in general the converse of this statement is not true , for instance see the example given bellow.

Example 2.4.6. Let $M(X) = t + \frac{1}{t}$ $\forall t \in X$ be a self map on $X = [0, \infty)$ with metric d such that,

$$\begin{aligned}
d(M(t_1), M(t_2)) &= d\left(t_1 + \frac{1}{t_1}, t_2 + \frac{1}{t_2}\right) \\
&= \mid t_1 + \frac{1}{t_1} - t_2 - \frac{1}{t_2} \mid \\
&= \mid t_1 - t_2 + (\frac{1}{t_1} - \frac{1}{t_2}) \mid \\
&= \mid (t_1 - t_2) + (\frac{t_2 - t_1}{t_1 t_2}) \mid
\end{aligned}$$

$$= \mid (t_1 - t_2) - (\frac{t_1 - t_2}{t_1 t_2}) \mid$$

$$= \mid t_1 - t_2 \mid \mid 1 - \frac{1}{t_1 t_2} \mid$$

$$< \mid t_1 - t_2 \mid$$

$$= d(t_1, t_2).$$

this shows that M is contractive but its not a contraction.

Definition 2.4.7. Non-expensive Mapping

A self map $M : X \mapsto X$ on a metric space is a Non-expensive Mapping if,

$$d(M(\eta_1), M(\eta_2)) \leq d(\eta_1, \eta_2); \quad \forall \eta_1, \eta_2 \in X \quad \eta_1 \neq \eta_2$$

Note: "Every contractive mapping is a non-expensive mapping but every non-expensive mapping need not be contractive mapping and hence is not a contraction". For example identity map is non-expensive but not a contraction.

2.5 Banach Contraction Principle

The most important and basic result of fixed point theory is Banach fixed point theorem [8] or contraction principle. A polished Mathematician Stefan Banach first present Banach Contraction Principle in his PHD research during 1922.

Theorem 2.5.1.

Let M be a self map on a metric space (X, d), suppose $0 \leq \alpha < 1$ such that,

$$d(M(\eta_1), M(\eta_2)) \leq \alpha d(\eta_1, \eta_2); \quad \forall \eta_1, \eta_2 \in X \quad \eta_1 \neq \eta_2$$

then there is a unique fixed point of M.

Example 2.5.2. A $M : (\mathbb{R}, d) \mapsto (\mathbb{R}, d); X = \mathbb{R}$ is such that $t \longrightarrow 1 + \frac{u}{4}$ then,

$$
\begin{aligned}
d(M(u), M(v)) &= d(1 + \frac{u}{4}, 1 + \frac{v}{4}); \quad \forall u, v \in \mathbb{R} \\
&= \frac{1}{4} \mid u - v \mid \\
&\leq \lambda d(u, v)
\end{aligned}
$$

since it satisfied all condition of the statement of Theorem 2.5.1 with unique fixed point $u = \frac{4}{3}$.

Example 2.5.3. Suppose $X = (0, 1] \subset \mathbb{R}$ and define a self map M on X such that $t \longrightarrow \frac{1}{4}u, \forall \ t \in X$ then,

$$
\begin{aligned}
d(M(u), M(v)) &= d(\frac{1}{4}u, \frac{1}{4}v); \quad \forall u, v \in \mathbb{R} \\
&= \frac{1}{4} \mid u - v \mid \\
&\leq \lambda d(u, v).
\end{aligned}
$$

Here $\lambda = \frac{1}{4}$ but M is not complete and has no fixed point which is contradiction of 2.5.1, this shows that for sure existance of 2.5.1 completeness is necessary.

2.6   Fuzzy Mappings

This section deals with the definition and examples related to fuzzy mappings.

Definition 2.6.1. (Fuzzy Sets)

Let X be a nonempty set, a fuzzy set F in X is defined as its membership function $\mu_F(u)$ consists of degree or grade of membership of element u in fuzzy set F for all $u \in X$. It is clear that fuzzy set F is represented in the form of ordered pairs *e.g,*

$$
F = \{(u, \mu_F(u))/u \in X\}
$$

Membership function for a fuzzy set F on X is defined as,

$$
\mu_F : X \mapsto [0, 1]
$$

that is each element of X is mapped to values between 0 and 1, these values are called degree or grade of membership. Fuzzy sets are graphically represented with the help of membership function.
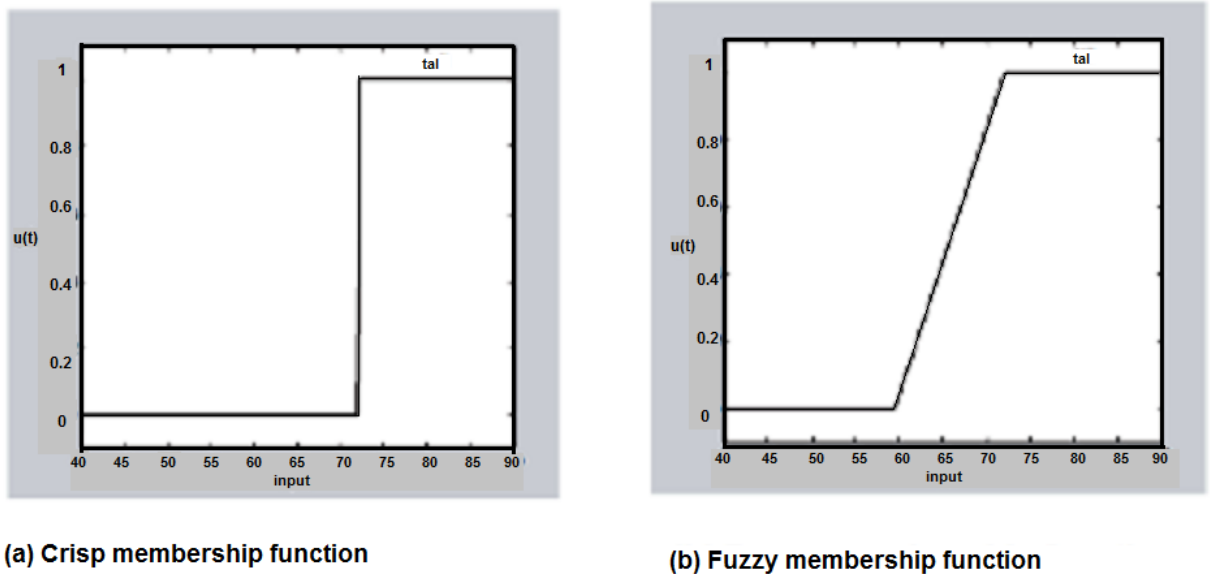

(a) Crisp membership function

(b) Fuzzy membership function

FIGURE 2.4: Fuzzy membership functions

Fuzzy Set Operations

- **OR**

$$\mu_{A \cup B}(t) = \max[\mu_A(t), \mu_B(t)] \quad \mu_{A \cup B}(t) = \mu_A(t) + \mu_B$$

- **AND**

$$\mu_{A \cap B}(t) = \min[\mu_A(t), \mu_B(t)] \quad \mu_{A \cup B}(t) = \mu_A(t)\mu_B$$

- **NOT**

$$\mu_{\overline{A}} = 1 - \mu_A(t)$$

Definition 2.6.2. (Fuzzy Number)

A fuzzy number is a number whose membership function is partially continuous

and has value,

$$\mu_F(x) = 1$$

Following graphs show the fuzzy number x, fuzzy number near x(neighborhood of x) and fuzzy number almost x respectively.
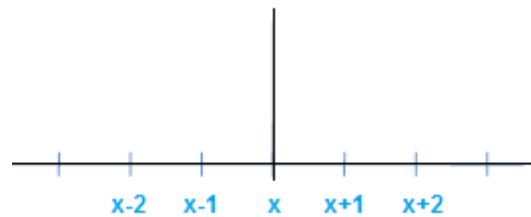

FIGURE 2.5: Fuzzy Number **x**
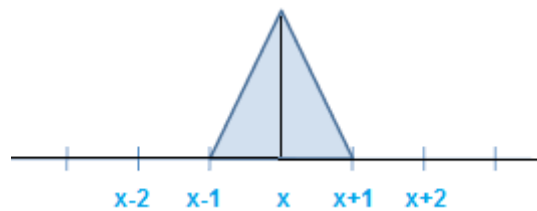
FIGURE 2.6: Fuzzy Number **near x**
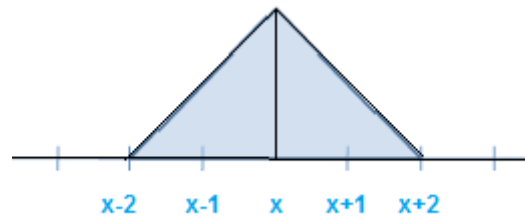
FIGURE 2.7: Fuzzy Number **almost x**

Definition 2.6.3. (Fuzzy Mappings)

Let X be any set and Y be any metric linear space. M is called a fuzzy mapping iff M is a mapping from X into $\mathcal{W}(Y)$ that is $M(x) \in \mathcal{W}(Y)$ for each $x \in X$. Here $\mathcal{W}(Y)$ is a collection of family of fuzzy sets.

In a mathematical model which is difficult to derive, fuzzy system is suitable for uncertainty or approximation. Under incomplete information fuzzy logic helps to make decision.

Example 2.6.4. Fuzzy Cognitive Maps are one of the most familiar example of Fuzzy mappings. These maps are introduced by a political scientist Kosko [26] in 1970s to represent social scientific knowledge. Generally Cognitive maps are the casual relationships of the concepts which are designed in the graphs. The graphical representation of these facts of a given framework is considered as a fuzzy cognitive map(FCM). Simply the collaboration of cognitive mapping and fuzzy logic is known as fuzzy cognitive mapping. The basic tool for cognitive is **'Theory of graphs'**. Most of the calculations are based on graph theory. The important component of a mapped structure collectively make the fuzzy cognitive map(FCM). To calculate end results and in order to run simulations FCM is is used as an important tool. In political science, military science, history,international relations, where meta-system language and system concepts both are fundamentally fuzzy, FCM can be easily applicable. Distributed intelligence can also be represented by fuzzy cognitive maps, the figure [39] given bellow represents casual relationship of soft knowledge.
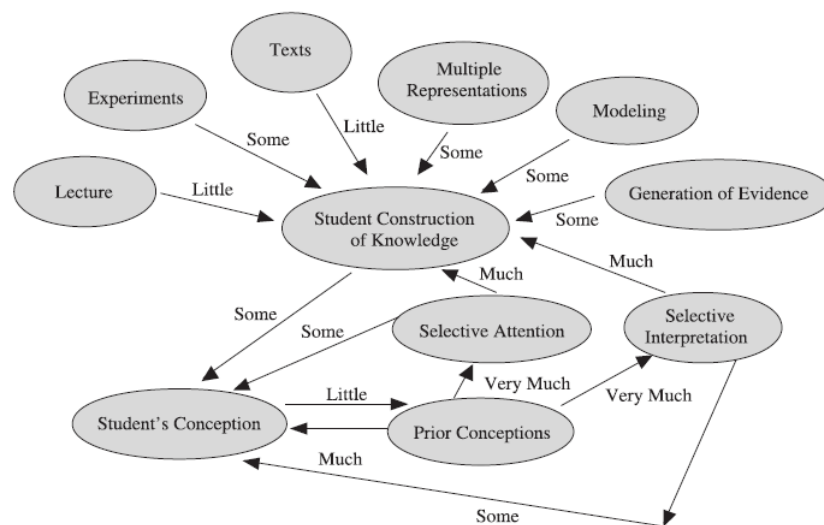

FIGURE 2.8: Fuzzy Cognitive Map

Some other examples of Fuzzy Cognitive Maps are as follows,

Example 2.6.5. Following figure with its matrix form make easy to understand the simplest form of fuzzy cognitive map:-
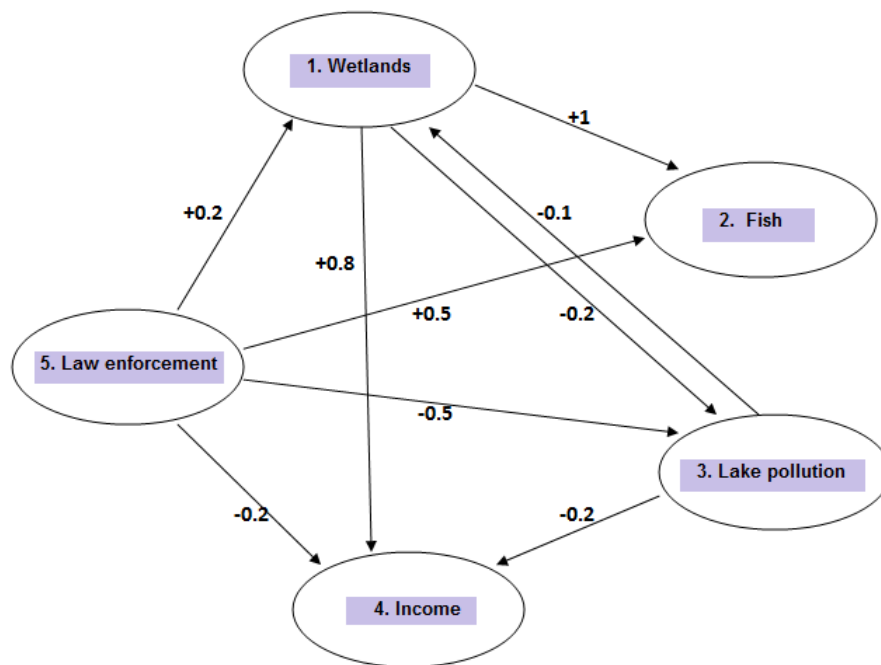
FIGURE 2.9: FCM given in ozesmi's paper [35] presented in 2004

Tab. 1: Representing the Fuzzy Cognitive Map along its matrix form given in above figure.

Groups zesmi zesmi 2004	*Wetlands*	*Fish*	*LakePln.*	*Inc.*	*LawEnft.*
Wetlands	0.0	1.0	-0.1	0.8	0.0
Fish	0.0	0.0	0.0	0.0	0.0
LakePln.	-0.2	0.0	0.0	-0.2	0.0
Inc.	0.0	0.0	0.0	0.0	0.0
LawEnft.	0.2	0.5	-0.5	-0.2	0.0
Outdegree	1.90	0.00	0.40	0.00	1.40
Indegree	0.40	1.50	0.60	1.20	0.00
Pitch	2.30	1.50	1.00	1.20	1.40
Type	*Ord.*	*Rcvr.*	*Ord.*	*Rcvr.*	*Transmitter*
Dnsty.	*TotalFact.*	*TotalConnections*	*Nr.Transmitter*	*Nr.Rcvr.*	*Nr.Ord.*
0.36	5	9	1	2	2

The above figure and table is taken from milti-step cognitive maps by Ozesmi [35], the transmitted variable is law enforcement. In this map, the outdegree of laws enforcement is 1.5 while ist indegree is 0. All other variables are affected by law enforcement but the other variables have no influence on it. The reciever variable in this map is income which is variable 3, the outdegree of income is 0 while its input is 1.20. Therefore, in this map there is an influence of other variables on income. The fish population is followed by wetland and income so it is the central

variable in this map.

Example 2.6.6. Fuzzy cognitive mapping for bad weather driving is given bellow. This figure is taken out from Bart Kosko's book [54] named as "Fuzzy Thinking".
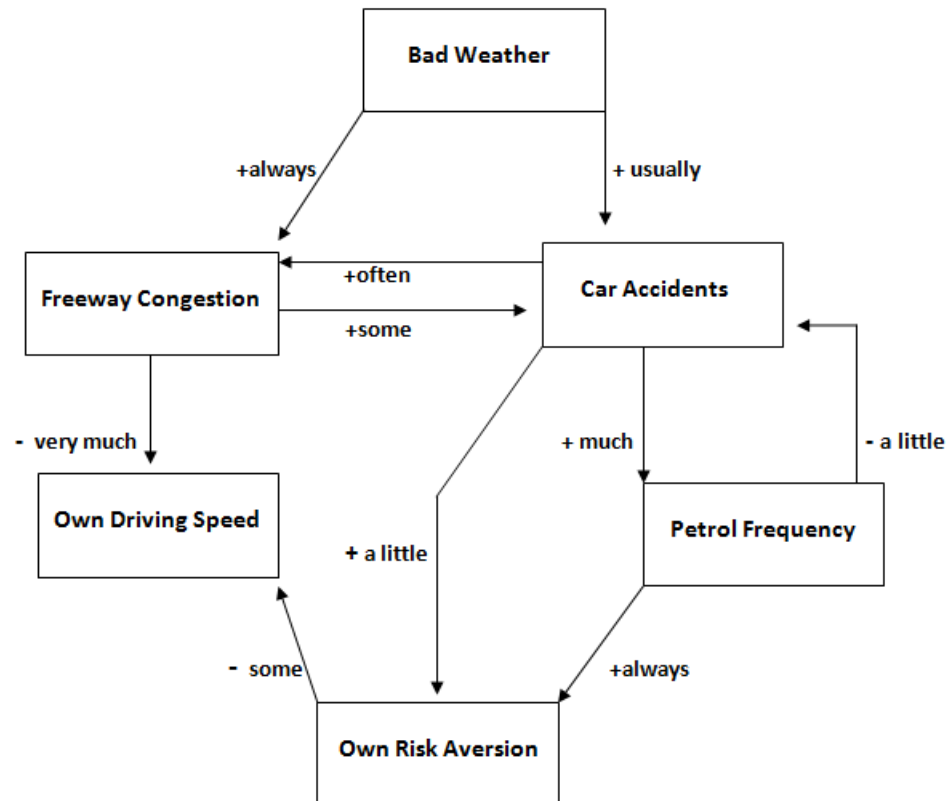
FIGURE 2.10: FCM for Bad weather driving

Above figure shows the effect of bad weather on driving on a Clifornia freeway in daytime. The symbols + and − indicates the types of relationship between the factors and the casual relationship is defined with the words "usually" and "a little". The table bellow shows the values of bad weather from 20-80. We can see that after 10,12 and 13 steps there is a convergence for the bad weather levels in the simulations of 20, 30 and 40.

Factor	B.Weather=20	B.Weather=30	B.Weather=40
B.Weather	20	30	40
Freeway Cong.	21	31	41
Accidents	11	16	28
Own R. Aver.	85	86	88
Patrol Freq.	100	100	100
Own Driv.Sp.	100	100	100

Similarly, we can see that table for simulations of 60, 70 and 80. The convergence will take place after 12, 12 and 10 steps.

Factor	B.Weather=60	B.Weather=70	B.Weather=80
B.Weather	60	70	80
Freeway Cong.	60	70	80
Accidents	72	84	89
Own R.Aver.	12	14	15
Patrol Freq.	0	0	0
Own Driv.Sp.	0	0	0

Chapter 3

Fuzzy Mappings in metric Space

In 1965 Zadeh [55] introduced the concept of fuzzy sets. Later on, in 1981 the concept of fuzzy mappings was inaugurated by Heilpern [22]. He also proved fixed point theorem which is an extension of the Banach contraction principle. Abu-Donia [1] also proved fixed point results for fuzzy mappings. These theorems are helpful in geometrical problems related to physics, but needs some extension. Although, the proofs of the main consequence of Abu-Donia's theorems are not correct. In 2008 T. Kamran [23] presented the correct results of these proofs in his paper. In this Chapter we review the results presented in [23].

3.1 Notations

Through out this Chapter:
(X, d) is a metric space, D the distance between the sets. Let P and Q be the non-empty subsets of X then,

$$D(P, Q) = \inf\{d(p, q) : p \in P, q \in Q\}$$

$C(X)$ denotes the set of all nonempty compact subsets of X, $CB(X)$ represents the set of all nonempty bounded closed subsets of X. Here H denotes the Hausdorff

metric with respect to d, that is,

$$H(P,Q) = \max\{\sup_{x \in P} D(\{x\}, Q), \sup_{y \in Q} D(\{y\}, P)\}$$

Let T maps form X into $CB(X)$ be a setvalued mapping defined as, for every $x \in X, Tx \subset X$. A point $p \in X$ is called a fixed point of a multivalued map T iff $p \in Tp$.

Here by Tp we mean $T(p)$. Moreover, we wrote $D(\{x\}, P)$ as $D(x, P)$ and $T(x)$ as Tx.

Let us suppose that: $K(X) = \{\eta \in I^X : \widehat{\eta} \in CB(X)\}$, where $\widehat{\eta} = \{x \in X : \eta(x) = \max_{t \in X} \eta(t)\}$ and $\pi : K(X) \mapsto CB(X)$ and $\pi(\eta) = \widehat{\eta}$.

Definition 3.1.1. A set valued mapping M on X into $K(X)$ is called a fuzzy mapping on X, $\widehat{M}$ denotes the composition of η and M, that is

$$\eta \circ M = \widehat{M} = \{y \in X : M_x y = \max_{z \in X} M_x z\} \tag{3.1}$$

Let $M : X \mapsto K(X)$ be a fuzzy mapping and $v \in X$ is a fixed point of M if $M_v v \geq M_v x \quad \forall x \in X$.

Lemma 3.1.2. A point $v \in X$ is a fixed point for a fuzzy mapping $M : X \mapsto K(X)$, iff v is a fixed point for the induced mapping $\widehat{K}$ mapped from X into $CB(X)$.

Proof. $v \in X$ is fixed point of $M : X \mapsto K(X)$,

$$\Leftrightarrow M_v(v) \geq M_v(t), \forall t \in X$$
$$\Leftrightarrow M_v(v) = \max_{t \in X} M_v(t)$$
$$\Leftrightarrow v \quad \text{is fixed point of} \quad \widehat{M} \quad \text{maps from X into CB(X).}$$

$\Leftrightarrow v$ is fixed point of $\widehat{M}$ maps from X into $CB(X)$. □

Lemma 3.1.3. let A be a nonempty subset of X in a metric space (X, d), then

$$D(u, A) \leq d(u, t) + D(t, A) \quad \text{for any} \quad u, t \in X$$

Proof. Suppose that $t \in X$ then,

$$\begin{aligned}
D(u, A) &= \inf\{d(u, v) : v \in A\} \\
&\leq \inf\{d(u, t) + d(t, v) : v \in A\} \\
&= d(u, t) + \inf\{d(t, v) : v \in A \\
&= d(u, t) + D(t, A).
\end{aligned}$$

$\square$

Lemma 3.1.4. Suppose that a non-decreasing function $\Psi : \mathbb{R}^+ \mapsto \mathbb{R}^+$ which satisfies that Ψ is continuous from right and

$$\sum_{m=0}^{\infty} \Psi^m(z) < \infty \quad \forall \quad z > 0$$

Where Ψ^m denotes the mth iterative function of Ψ. Then $\Psi(z) < z$.

3.2 Fixed Point Results in Metric Space

The followings are the main results presented by Abu-Donia[1].

Theorem 3.2.1.

Let (X, d) be a complete metric space and let M, N be two fuzzy mappings defined as $M, N : X \to K(X)$ the mappings induced by M, N are $\widehat{M}, \widehat{N} : X \mapsto CB(X)$. Let Ψ satisfying lemma 3.1.4.

$$H(\widehat{M_a}, \widehat{N_b}) \leq \Psi\left(\max\left(d(a, b), D(x, \widehat{M_a}), D(y, \widehat{N_b}), \frac{D(a, \widehat{N_b}) + D(b, \widehat{M_a})}{2}\right)\right)$$
$$(3.2)$$

Where $a, b \in X$, then there exist a common fixed point of M and N.

Theorem 3.2.2.

Let (X, d) be a complete metric space. Let $M_n : x \mapsto K(X)$ be a sequence of fuzzy mappings and by 3.1, $\widehat{M_n}$, a fuzzy mapping induced by M_n. Let Ψ satisfying the lemma 3.1.4 , suppose that $a, b \in X$ and i, j be the positive integers such that

$$H(\widehat{M_i}a, \widehat{M_j}b) \leq \Psi \left(\max \left(d(a,b), D(a, \widehat{M_i}a), D(b, \widehat{M_j}b), \frac{D(a, \widehat{M_j}b) + D(b, \widehat{M_i}a)}{2} \right) \right)$$
$$(3.3)$$

then there exist a common fixed point of M_n.

From proofs of above theorems it follows that if $\widehat{M}, \widehat{N}, M_n$ are the mappings from $X \mapsto CB(X)$ then for every $P \in CB(X), \exists$ an $p \in P \longrightarrow d(x,p) = D(x,P), \forall x \in X$. but it is true in case of P, a compact subset of X.

Example 3.2.3. [23]

"[4 page 480] Let ℓ_2 denote the Hilbert space of all square summable sequences of real numbers; let $a = (-1, -\frac{1}{2}, \cdots, -\frac{1}{n}, \cdots$ and; for each $n = 1, 2, \cdots$, let e_n be the vector in ℓ_2 with zeros in all its coordinates expect the nth coordinates which is equal to 1. Let $B = \{e_1, e_2, \cdots e_n, \cdots\}$. Since $\| a - e_n \| = \left(\| a \|^2 + 1 = \frac{2}{n} \right)^{\frac{1}{2}}$ for each $n = 1, 2, \cdots$, and there is no e_n in B such that $\| a - e_n \| \leq D(a, B)$."

T. Kamran [23] said that if $\widehat{M}, \widehat{N}, M_n$ are the mappings from $X \mapsto C(X)$ or the mappings from X into $PC(X)$, (where the set of nonempty closed and proximinal subsets is represented by $PC(X)$) then Abu-Donia's theorem will work. The Proximinal subsets were first introduced by [37].

Definition 3.2.4. Let V be the subset of X and if for each $x \in X$ there exist an element $v \in V$ such that,

$$d(x, v) = d(x, V)$$

then the subset V of X is called proxeminal.

T. Kamran replaced the inequalities by strict inequalities with the Lemma [25] given bellow.

Lemma 3.2.5. For each $p \in P$ there is a $q \in Q$ such that,

$$d(p,q) < \delta, \quad \text{where} \quad P, Q \in CB(X) \quad \text{with} \quad H(P,Q) < \delta$$

The new correction of the above theorems [23] is then given as follows;

Theorem 3.2.6.

Let (X, d) is a complete metric space and suppose that M and N mapped from X into $K(X)$ be two fuzzy mappings. According to the equation 3.1, induced mappings of M and N are $\widehat{M}, \widehat{N} : X \mapsto CB(X)$. Let Ψ satisfying Lemma 3.1.4.

$$H(\widehat{M}_s, \widehat{N}_t) < \Psi \left(\max \left(d(s,t), D(s, \widehat{M}_s), D(t, \widehat{N}_t), \frac{D(s, \widehat{N}_t) + D(t, \widehat{M}_s)}{2} \right) \right) \quad (3.4)$$

Where $s, t \in X$, Then there exist a fixed point of M and N.

Proof. Let s_0 be a point in X, Since $\widehat{M}_1 s_0$ is nonempty therefore there is a point $s_1 \in \widehat{M}_1 s_0$.

Let

$$\delta = \Psi \left(\max \left(d(s_0, s_1), D(s_0, \widehat{M}_1 s_0), D(s_1, \widehat{M}_2 s_1), \frac{D(s_0, \widehat{M}_2 s_1) + D(s_1, \widehat{M}_1 s_0)}{2} \right) \right)$$

then by above condition 3.4 we have

$$H(\widehat{M}_1 s_0) < \delta$$

by lemma 3.1.4, we have $s_2 \in X$ such that $s_2 \in \widehat{M}_2 s_1$ and

$d(s_1, s_2) < \delta$

$$= \Psi \left(\max \left(d(s_0, s_1), D(s_0, \widehat{M}_1 s_0), D(s_1, \widehat{M}_2 s_1), \frac{D(s_0, \widehat{M}_2 s_1) + D(s_1, \widehat{M}_1 s_0)}{2} \right) \right)$$

$$\leq \Psi \left(\max \left(d(s_0, s_1), d(s_0, s_1), d(s_1, s_2), \frac{d(s_0, s_2) + d(s_1, s_1)}{2} \right) \right)$$

$$= \Psi \left(\max \left(d(s_0, s_1), d(s_1, s_2), \frac{d(s_0, s_2)}{2} \right) \right)$$

$$\leq \Psi \left(\max \left(d(s_0, s_1), d(s_1, s_2), \frac{d(s_0, s_1 + d(s_1, s_2)}{2} \right) \right)$$

$$\leq \Psi(\max(d(s_0, s_1), d(s_1, s_2))) \tag{3.5}$$

Suppose that,

$$\max(d(s_0, s_1), d(s_1, s_2) = d(s_1, s_2)$$

then by using $\Psi(z) < z$ for $z > 0$, we have

$$d(s_1, s_2) < \Psi(d(s_1, s_2))$$

Which is a contradiction, this implies that,

$$\max(d(s_0, s_1), d(s_1, s_2)) = d(s_0, s_1)$$

by eq.3.5 we have,

$$d(s_1, s_2) < \Psi(d(s_0, s_1)) \tag{3.6}$$

in the same manner, we make sequence $\{s_n\} \quad \forall n \geq 1$ such that,

$$s_{2n-1} \in \widehat{M}_s 2n - 1, s_{2n} \in \widehat{N}_s 2n - 1, \tag{3.7}$$

and

$$d(s_{2n}, s_{2n+1}) < \Psi\left(d(s_{2n-1}, s_{2n})\right), \tag{3.8}$$

$$d(s_{2n-1}, s_{2n}) < \Psi\left(d(s_{2n-2}, s_{2n-1})\right), \tag{3.9}$$

$$\therefore d(s_{n+1}, s_n) < \Psi\left(d(s_{n-1}, s_n)\right) \leq \Psi^n(d(s_0, s_1)) \quad \forall n \geq 1. \tag{3.10}$$

Thus, for $m, n(n > m)$ positive integers, we have

$$d(s_m, s_n) \leq d(s_m, s_{m+1}) + \cdots + d(s_{n-1}, s_n)$$

$$< \Psi^m d(s_0, s_1) + \cdots + \Psi^{n-1} d(s_0, s_1)$$

$$= \sum_{r=m}^{n-1} \Psi^r d(s_0, s_1)$$

$$\leq \sum_{r=m}^{\infty} \Psi^r d(s_0, s_1)$$

According to the following condition;

$$\sum_{n=1}^{\infty} \Psi^n d(z) < \infty \quad for\ each \quad z > 0,$$

$\{s_n\}$ is a Cauchy sequence.
let $s_n \to p \in X$. By lemma 3.1.3

$$D(p, \widehat{N}_p) \leq d(p, s_{2n-1}) + D(s_{2n-1}, \widehat{N}_p) \leq d(p, s_{2n-1}) + H(\widehat{M}_{s_{2n-2}}, \widehat{N}_p)$$

$$< d(p, s_{2n-1}) + \Psi \left(\max \left(d(s_{2n-2}, p), D(s_{2n-2}, \widehat{M}_{s_{2n-2}}), D(p, \widehat{N}_p), \frac{D(s_{2n-2}, \widehat{N}_p) + D(p, \widehat{M}_{s_{2n-2}})}{2} \right) \right)$$

$$< d(p, s_{2n-1}) + \Psi \left(\max \left(d(s_{2n-2}, p), d(s_{2n-2}, s_{2n-1}), D(p, \widehat{N}_p), \frac{D(s_{2n-2}, \widehat{N}_p) + d(p, s_{2n-1})}{2} \right) \right)$$

By using continuity of Ψ we have,

$$D(p, \widehat{N}_p) \leq \Psi \left(D(p, \widehat{N}_p) \right) < D(p, \widehat{N}_p)$$

this contradicts and arise from $p \in \widehat{N}_p$ which is closeness of $\widehat{N}$,

in the same way $p \in \widehat{M}_p$.

So by lemma 3.1.2, M and N has a common fixed point that is p. $\qquad \square$

Theorem 3.2.7.

Let (X, d) be a complete metric space. Let $\{M_n\}$ from X into $K(X)$ be a sequence of fuzzy mappings and by eq.3.1, $\widehat{M_n}$, a fuzzy mapping induced by M_n. Let Ψ satisfying the lemma 3.1.4, suppose that $s, t \in X$ and i, j be the positive integers such that,

$$H(\widehat{M}_i s, \widehat{M}_j t) < \Psi \left(\max \left(d(s, t), D(s, \widehat{M}_i s), D(t, \widehat{M}_j t), \frac{D(s, \widehat{M}_j t) + D(t, \widehat{M}_i s)}{2} \right) \right)$$

$$(3.11)$$

then there exist a common fixed point of $\{M_n\}$.

Proof. Let $s_0 \in X$ be a point, there is a point $s_1 \in \widehat{M_1}s_0$ because $\widehat{M_1}s_0$ in nonempty. Let

$$\delta = \Psi\left(\max\left(d(s_0, s_1), D(s_0, \widehat{M_1}s_0), D(s_1, \widehat{M_2}s_1), \frac{D(s_0, \widehat{M_2}s_1) + D(s_1, \widehat{M_1}s_0)}{2}\right)\right)$$

by inequality 3.11 we have,

$$H(\widehat{M_1}s_0, \widehat{M_2}s_1) < \delta$$

by lemma 3.1.4, $s_2 \in X$ such that $s_2 \in \widehat{M_2}s_1$

and

$$
\begin{aligned}
d(s_1, s_2) < \delta &= \Psi\left(\max\left(d(s_0, s_1), D(s_0, \widehat{M_1}s_0), D(s_1, \widehat{M_2}s_1), \frac{D(s_0, \widehat{M_2}s_1) + D(s_1, \widehat{M_1}s_0)}{2}\right)\right) \\
&\leq \Psi\left(\max\left(d(s_0, s_1), d(s_0, s_1), d(s_1, s_2), \frac{d(s_0, s_2) + d(s_1, s_1)}{2}\right)\right) \\
&= \Psi\left(\max\left(d(s_0, s_1), d(s_1, s_2), \frac{d(s_0, s_2)}{2}\right)\right) \\
&\leq \Psi\left(\max\left(d(s_0, s_1), d(s_1, s_2), \frac{d(s_0, s_1) + d(s_1, s_1)}{2}\right)\right) \\
&\leq \Psi(\max(d(s_0, s_1), d(s_1, s_2))) \quad\quad\quad (3.12)
\end{aligned}
$$

Let us suppose that,

$$\max(d(s_0, s_1), d(s_1, s_2)) = d(s_1, s_2)$$

Therefore, by $\Psi(z) < z \quad \forall z > 0$ we have,

$$d(s_1, s_2) < \Psi(d(s_1, s_2)) < d(s_1, s_2)$$

This contradicts thus,

$$\max(d(s_0, s_1), d(s_1, s_2)) = d(s_0, s_1)$$

Using inequality 3.12, we have

$$d(s_1, s_2) < \Psi(d(s_0, s_1)) \tag{3.13}$$

In the same manner, we make a sequence $\{s_n\}$ $\forall n \geq 1$, such that

$$s_n \in \widehat{M_n} s_{n-1} \tag{3.14}$$

$$d(s_{n+1}, s_n) < \Psi(d(s_{n-1}, s_n)) < \Psi^n(d(s_0, s_1)) \quad \text{for all} \quad n \geq 1 \tag{3.15}$$

So $\{s_n\}$ is a Cauchy sequence in X same as in last theorem.
Suppose that $s_n \longrightarrow p \in X$, for $m \in \mathbb{N}$ by 3.11,

$$
\begin{aligned}
D(p, \widehat{M_n}p) &\leq d(p, s_m) + D(s_m, \widehat{M_n}p) \\
&\leq d(p, s_m) + H(\widehat{M_m}s_{m-1}, \widehat{M_n}p) \\
&< d(p, s_m) + \Psi\left(\max\left(d(s_{m-1}, p), D(s_{m-1}, \widehat{M_m}s_{m-1}), D(p, \widehat{M_n}p), \frac{D(s_{m-1}, \widehat{M_n}p) + D(p, \widehat{M_m}s_{m-1})}{2} \right) \right) \\
&\leq d(p, s_{m-1}) + \Psi\left(\max\left(d(s_{m-1}, p), d(s_{m-1}, s_m), D(p, \widehat{M_n}p), \frac{D(p_{m-1}, \widehat{M_n}p) + d(p, s_m)}{2} \right) \right)
\end{aligned}
$$

Let $n \longrightarrow \infty$ and by the continuity condition of Ψ, we have,

$$D(p, \widehat{M_n}p) \leq \Psi(D(p, \widehat{M_n}p)) < D(p, \widehat{M_n}p),$$

This contradicts and it arises from $p \in \widehat{M_n}$ $\forall n = 1, 2, 3, \cdots$ which is closeness of $\widehat{M_n}$. Therefore, by lemma 3.1.2 there is a common fixed point of M_n $\forall n = 1, 2, 3, \cdots$. $\square$

Definition 3.2.8. [1]

"Let P and Q be two self mappings of a metric space (X, d) then these mappings are compatible if

$$\lim_{n \to \infty} d(PQt_n, QPt_n) = 0$$

here $\{t_n\} \in X$ be a sequence and

$$\lim_{n \to \infty} Pt_n = \lim_{n \to \infty} Pt_n = u \quad \text{for some} \quad u \subset X$$

Theorem 3.2.9.

Let M and N mapped from X into $K(X)$ be two fuzzy mappings of a complete metric space (X, d). Let the induced mappings of M and N are $\widehat{M}, \widehat{N} : X \mapsto CB(X)$ then according to equation 3.1,

(i) There exists a sequence $\{s_n\} \in X$ implies that,

$$\lim_{n \to \infty} M_{s_n} = \lim_{n \to \infty} N_{s_n} = u \quad \text{for some} \quad u \subset X,$$

(ii) $H(\widehat{M}_s, \widehat{M}_t) < \max\{H(\widehat{N}_s, \widehat{N}_t), \frac{1}{2}[H(\widehat{M}_s, \widehat{N}_s) + H(\widehat{M}_t, \widehat{N}_t)],$
   $\frac{1}{2}[H(\widehat{M}_t, \widehat{N}_s) + H(\widehat{M}_s, \widehat{N}_t)]\} \quad \forall s, t \in X, s \neq t$

(iii) $\widehat{M}X \subset \widehat{N}X$

If one of $\widehat{M}X$ and $\widehat{N}X$ is a complete subspace of X this implies that there is a unique fixed point of M and N.

Proof. As (i) is satisfied by $\widehat{M}X$ and $\widehat{N}X$, then there exists a sequence $\{s_n\} \in X$ implies that,

$$\lim_{n \to \infty} M_{s_n} = \lim_{n \to \infty} N_{s_n} = u \quad \text{for some} \quad u \subset X,$$

Let us suppose that $\widehat{N}X$ is complete, this implies that for $r \in X, \lim_{n \to \infty} \widehat{N}_{s_n} = \widehat{N}_r$ and $\lim_{n \to \infty} \widehat{M}_{s_n} = \widehat{N}_r$.
Now let $\widehat{M}_r \neq \widehat{N}_r$ then $(ii) \Rightarrow$

$$H(\widehat{M}_{s_n}, \widehat{M}_r) < \max\left\{H(\widehat{N}_{s_n}, \widehat{N}_r), \frac{1}{2}[H(\widehat{M}_{s_n}, \widehat{N}_{s_n}) + H(\widehat{M}_r, \widehat{N}_r)], \frac{1}{2}[H(\widehat{M}_r, \widehat{N}_{s_n}) + H(\widehat{M}_{s_n}, \widehat{N}_r)]\right\}$$

As $n \longrightarrow \infty \Rightarrow,$

$$H(\widehat{N}_r, \widehat{M}_r) \leq \max\left\{H(\widehat{N}_r, \widehat{N}_r), \frac{1}{2}[H(\widehat{N}_r, \widehat{N}_r) + H(\widehat{M}_r, \widehat{N}_r)], \frac{1}{2}[H(\widehat{M}_r, \widehat{N}_r) + H(\widehat{N}_r, \widehat{N}_r)]\right\}$$
$$\leq \frac{H(\widehat{M}_r, \widehat{N}_r)}{2}$$

This contradicts our supposition that $\widehat{M}_r \neq \widehat{N}_r$ therefore $\widehat{M}_r = \widehat{N}_r$,
since $\widehat{M}$ and $\widehat{N}$ are weakly compatible therefore $\widehat{M}\widehat{M}_r = \widehat{M}\widehat{N}_r$ this implies that

$\widehat{M}\widehat{M_r} = \widehat{N}\widehat{M_r} = \widehat{M}\widehat{N_r} = \widehat{N}\widehat{N_r}$

Finally, to prove $\widehat{M}$ and $\widehat{N}$ has a common fixed point r we suppose that $\widehat{M_r} \neq \widehat{M}\widehat{M_r}$ then,

$$H(\widehat{M_r}, \widehat{M}\widehat{M_r}) \leq \max\left\{ H(\widehat{N_r}, \widehat{N}\widehat{M_r}), \tfrac{1}{2}[H(\widehat{M_r}, \widehat{N_r}) + H(\widehat{M}\widehat{M_r}, \widehat{N}\widehat{M_r})], \tfrac{1}{2}[H(\widehat{M}\widehat{M_r}, \widehat{N_r}) + H(\widehat{M_r}, \widehat{N}\widehat{M_r})]\right\}$$

$$\leq \max\left\{ H(\widehat{M_r}, \widehat{M}\widehat{M_r}), H(\widehat{M}\widehat{M_r}, \widehat{M_r})\right\}$$

$$= H(\widehat{M_r}, \widehat{M}\widehat{M_r}),$$

this contradicts, hence $\widehat{M_r} = \widehat{M}\widehat{M_r}$ and $\widehat{N}\widehat{M_r} = \widehat{M}\widehat{M_r} = \widehat{M_r}$, Similarly for $\widehat{M}X$ a complete subspace of X as $\widehat{M}X \subset \widehat{N}X$.

Since r is a common fixed point of $\widehat{M}$ and $\widehat{N}$ therefore, according to the lemma 3.1.2 r is fixed point of M and N.

'Uniqueness of fixed point follows easily'. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Chapter 4

Fuzzy mappings in b-Metric Spaces

In the previous chapter, first we studied Abu Donia's [1] fixed point theorems in metric spaces then we focused on paper presented by Kamran [23] for the corrected theorems in [1].

The main purpose of this chapter is to extend fixed point theorem for fuzzy mappings in metric spaces to fixed point theorems for fuzzy mappings in b- metric spaces.

4.1 Notations

Through out this Chapter:

(X, d_b) is a b- metric space. The distance $D_b(P, Q)$ between the subsets then, P, Q of X is then given by

$$D_b(P, Q) = \inf\{d_b(p, q) : p \in P, q \in Q\}.$$

$C(X)$ denotes the set of all nonempty compact subsets of X and $CB(X)$ represents the set of all nonempty bounded closed subsets of X. The Hausdorff metric with

respect to d_b is denoted by H_b is given as

$$H_b(U, V) = \max(\sup_{x \in U} D_b(\{x\}, V), \sup_{y \in V} D_b(\{y\}, U))$$

Let T maps from X into $CB(X)$ be a setvalued mapping defined as, for every $x \in X, Tx \subset X$. A point $p \in X$ is called a fixed point of a multivalued map T iff $p \in Tp$. Here by Tp we mean $T(p)$.

Moreover, we write $D_b(\{x\}, U)$ as $D_b(x, U)$ and $T(x)$ as Tx.

Let us suppose that: $K_b(X) = \{\eta \in I^X : \widehat{\eta} \in CB(X)\}$, where $\widehat{\eta} = \{x \in X : \eta(x) = \max_{t \in X} \eta(t)\}$ and $\pi_b : K_b(X) \mapsto CB(X)$ and $\pi_b(\eta) = \widehat{\eta}$. The notion distance of a point x from a set U can be extended naturally for b-metric space as follows;

Definition 4.1.1. Let (X, d_b) be a b-metric space and a set U is such that, $\psi \neq U \subset X$ then

$$d_b(x, U) = \inf\{d_b(x, u) : u \in U\}.$$

Lemma 4.1.2. Let A be a nonempty subset of X and (X, d_b) be a b- metric space. Then,

$$D_b(u, A) \leq b\left(d_b(u, t) + D_b(t, A)\right) \quad \forall u, t \in X \quad \text{and} \quad b \geq 1.$$

Proof. Suppose that $t \in X$ then,

$$\begin{aligned}
D_b(u, A) &= \inf\{d_b(u, v) : v \in A\} \\
&\leq \inf\{b(d_b(u, t) + d_b(t, v)) : b \geq 1, v \in A \\
&= b(d_b(u, t) + \inf\{d_b(t, v) : v \in A\}) \\
&= b(d_b(u, t) + D_b(t, A)).
\end{aligned}$$

$\square$

Lemma 4.1.3. For each $u \in U$ there is a $v \in V$ such that,

$$d_b(u, v) < \delta, \quad \text{where} \quad U, V \in CB(X) \quad \text{with} \quad H_b(U, V) < \delta$$

4.2   Fixed Point Results in b-metric Space

Now the extended form of fixed point theorems in b-metric space is as follows:

Theorem 4.2.1.

Let (X, d_b) be a complete b-metric space with a continuous $b-$metric d_b and $b \geq 1$. Let M, N be two fuzzy mappings defined as $M, N : X \to K_b(X)$ the set-valued mappings induced by M, N are $\widehat{M}, \widehat{N} : X \to CB(X)$. Let Ψ be a function that satisfies Lemma 3.1.4. Moreover for any $s, t(s \neq t)$ in X

$$H_b(\widehat{M}_s, \widehat{N}_t) < \Psi \left(\max \left(d_b(s,t), D_b(s, \widehat{M}_s), D_b(t, \widehat{N}_t), \frac{D_b(s, \widehat{N}_t) + D_b(t, \widehat{M}_s)}{2} \right) \right)$$
(4.1)

there exist a common fixed point of M and N.

Proof. Let s_0 be a point in X, Since $\widehat{M}_{s_0}$ is nonempty therefore there is a point $s_1 \in \widehat{N}_{s_0}$. Let

$$\delta_b = \Psi \left(\max \left(d_b(s_0, s_1), D_b(s_0, \widehat{M}_{s_0}), D_b(s_1, \widehat{N}_{s_1}), \frac{D_b(s_0, \widehat{N}_{s_1}) + D_b(s_1, \widehat{M}_{s_0})}{2} \right) \right)$$
(4.2)

then by above Condition (4.2) we have,

$$H_b(\widehat{M}_{s_0}, \widehat{N}_{s_1}) < \delta_b$$

by Lemma 4.1.3 $\exists s_2 \in X$ such that, $s_2 \in \widehat{N}_{s_1}$ and

$$d_b(s_1, s_2) < H_b(\widehat{M}_{s_0}, \widehat{N}_{s_1})$$

$$d_b(s_1, s_2) < \delta_b = \Psi \left(\max \left(d_b(s_0, s_1), D_b(s_0, \widehat{M}_{s_0}), D_b(s_1, \widehat{M}_{s_1}), \frac{D_b(s_0, \widehat{M}_{s_1}) + D_b(s_1, \widehat{M}_{s_0})}{2} \right) \right)$$

$$\leq \Psi \left(\max \left(d(s_0, s_1), d_b(s_0, s_1), d_b(s_1, s_2), \frac{D_b(x_0, s_2) + D_b(s_1, s_1)}{2} \right) \right)$$

$$= \Psi \left(\max \left(d_b(s_0, s_1), d(s_1, s_2), \frac{d_b(s_0, s_2)}{2} \right) \right)$$

$$\leq \Psi\left(\max\left(d_b(s_0, s_1), d_b(s_1, s_2), b\frac{d_b(s_0, s_1) + d_b(s_1, s_2)}{2}\right)\right) \tag{4.3}$$

Case-i For $b = 1$

It is a case of metric space directly follows from [23].

Case-ii For $b > 1$

Suppose that,

$$\max\left(d_b(s_0, s_1), d_b(s_1, s_2), b\frac{d_b(s_0, s_1) + d_b(s_1, s_2)}{2}\right) = d_b(s_1, s_2)$$

by using the condition $\psi(z) < z, \quad \forall z > 0$, so we have,

$$d_b(s_1, s_2) < \psi\left(d_b(s_1, s_2)\right) < d_b(s_1, s_2)$$

which is contradiction of our supposition.

Therefore (4.3) implies that,

$$d_b(s_1, s_2) \leq \Psi\left(\max\left(d_b(s_0, s_1), b\frac{d_b(s_0, s_1) + d_b(s_1, s_2)}{2}\right)\right) \tag{4.4}$$

Now if

$$\max\left(d_b(s_0, s_1), b.\frac{d_b(s_0, s_1) + d_b(s_1, s_2)}{2}\right) = d_b(s_0, s_1)$$

by using the condition $\psi(z) < z, \quad \forall z > 0$, (4.4) implies that

$$d_b(s_1, s_2) \leq \psi d_b(s_0, s_1) < d_b(s_0, s_1) \tag{4.5}$$

$$\text{if} \quad \max\left(d_b(s_0, s_1), b\frac{d_b(s_0, s_1) + d_b(s_1, s_2)}{2}\right) = b\frac{d_b(s_0, s_1) + d_b(s_1, s_2)}{2}$$

by using the condition $\psi(z) < z, \quad \forall z > 0$, 4.4 implies that

$$d_b(s_1, s_2) \leq \psi\left(b\frac{d_b(s_0, s_1) + d_b(s_1, s_2)}{2}\right)$$
$$< b\frac{d_b(s_0, s_1) + d_b(s_1, s_2)}{2}$$

$$\Rightarrow d_b(s_1, s_2) < \frac{b}{2}(d_b(s_0, s_1) + d_b(s_1, s_2))$$

$$\Rightarrow (1 - \frac{b}{2})d_b(s_1, s_2) < \frac{b}{2}d_b(s_0, s_1)$$

$$\Rightarrow (\frac{2 - b}{2})d_b(s_1, s_2) < \frac{b}{2}d_b(s_0, s_1)$$

$$\Rightarrow (2 - b)d_b(s_1, s_2) < bd_b(s_0, s_1)$$

Also

$$d_b(s_1, s_2) < \frac{b}{2 - b}d_b(s_0, s_1) \tag{4.6}$$

From both (4.5) and (4.6), we can conclude that

$$d_b(s_1, s_2) < bd_b(s_0, s_1) \tag{4.7}$$

$$\therefore d_b(s_1, s_2) < \psi(bd_b(s_0, s_1))$$

in the same manner, we make sequence $\{s_n\}, \forall n \geq 1$ such that,

$$s_{2n-1} \in \widehat{M}s_{2n-2}, s_{2n} \in \widehat{N}s_{2n-1}, \tag{4.8}$$

$$\text{and} \quad d_b(s_{2n}, s_{2n+1}) < \Psi\left(bd_b(s_{2n-1}, s_{2n})\right), \tag{4.9}$$

$$d_b(s_{2n-1}, s_{2n}) < \Psi\left(bd_b(s_{2n-2}, s_{2n-1})\right), \tag{4.10}$$

$$\therefore d_b(s_{n+1}, s_n) < \Psi\left(bd_b(s_{n-1}, s_n)\right) \leq \Psi^n(bd_b(s_0, s_1)) \quad \forall n \geq 1. \tag{4.11}$$

Thus, for $m, n(n > m)$ positive integers, we have

$$d_b(s_m, s_n) \leq d_b(s_m, s_{m+1}) + \cdots + d_b(s_{n-1}, a_n)$$

$$< \Psi^m(bd_b(s_0, s_1)) + \cdots + \Psi^{n-1}(bd_b(s_0, s_1))$$

$$= \sum_{r=m}^{n-1} \Psi^r(bd_b(s_0, s_1))$$

$$\leq \sum_{r=m}^{\infty} \Psi^r(bd_b(s_0, s_1))$$

According to the following condition;

$$\sum_{n=1}^{\infty} \Psi^n d(z) < \infty \quad \text{for each} \quad z > 0,$$

$\{s_n\}$ is a Cauchy sequence.
let $s_n \to p \in X$. By Lemma 4.1.2

$$D_b(p, \widehat{N}p) \le b\left(d_b(p, s_{2n-1}), D_b(s_{2n-1}, \widehat{N}p)\right) \le b\left(d_b(p, s_{2n-1}) + H_b(\widehat{M}s_{2n-2}, \widehat{N}p)\right)$$

$$< b\left(d_b(p, s_{2n-1}) + \Psi\left(\max\left(d_b(s_{2n-2}, p), D_b(s_{2n-2}, \widehat{M}s_{2n-2}), D_b(p, \widehat{N}p), \frac{D_b(s_{2n-2}, \widehat{N}p) + D_b(p, \widehat{M}s_{2n-2})}{2}\right)\right)\right)$$

$$< b\left(d_b(p, s_{2n-1}) + \Psi\left(\max\left(d_b(s_{2n-2}, p), d_b(s_{2n-1}, s_{2n-1}), D_b(p, \widehat{N}p), \frac{D_b(s_{2n-2}, \widehat{N}p) + d_b(p, s_{2n-1})}{2}\right)\right)\right)$$

$$< b\left(d_b(p, s_{2n-1}) + \Psi\left(\max\left(d_b(s_{2n-2}, p), d_b(s_{2n-2}, s_{2n-1}), D_b(p, \widehat{N}p), \frac{b(d_b(s_n, p) + D_b(p, \widehat{N}p)) + d_b(p, s_{2n-1})}{2}\right)\right)\right)$$

$\therefore$ by continuity of ψ there are following four possibilities,

$$D_b(p, \widehat{N}p) \le b\left(d_b(p, s_{2n-1}) + d_b(s_{2n-2}, p)\right), \tag{4.12}$$

$$D_b(p, \widehat{N}p) \le b\left(d_b(p, s_{2n-1}) + d_b(s_{2n-2}, s_{2n-1})\right), \tag{4.13}$$

$$D_b(p, \widehat{N}p) \le b\left(d_b(p, s_{2n-1}) + D_b(p, \widehat{N}p)\right)$$

$$\Rightarrow (1-b)D_b(p, \widehat{N}p) \le b.d_b(p, s_{2n-1})$$

$$\Rightarrow D_b(p, \widehat{N}p) \le \frac{b}{1-b}d_b(p, s_{2n-1}) \tag{4.14}$$

and $D_b(p, \widehat{N}p) \le b\left(d_b(p, s_{2n-1}) + \frac{b}{2}(d_b(s_n, p) + D_b(p, \widehat{N}p)) + \frac{1}{2}d_b(p, s_{2n-1})\right)$

$$(1 - \frac{b^2}{2})D_b(p, \widehat{N}p) \le (b + \frac{b}{2})d_b(p, s_{2n-1}) + (\frac{b^2}{2})d_b(s_n, p)$$

$$\Rightarrow D_b(p, \widehat{N}p) \le \frac{b}{2-b^2}(3d_b(p, s_{2n-1}) + b.d_b(s_n, p)) \tag{4.15}$$

Since $s_n \longrightarrow p$ as $n \longrightarrow \infty$

$\therefore$ from above Inequalities (4.12),(4.13),(4.14) and (4.15),

$$D_b(p, \widehat{N}p)) = 0, \Rightarrow p \in \widehat{N}p$$

Hence by Lemma 3.1.2, M and N has a common fixed point that is p. $\qquad \square$

Remark 4.2.2. Theorem 3.2.6 becomes a special case of Theorem 4.2.1 by taking $b = 1$.

Theorem 4.2.3.

Let (X, d_b) be a complete b-metric space with a continuous $b-$metric d_b and $b \geq 1$. Let $\{M_n\}$ from X into $K_b(X)$ be a sequence of fuzzy mappings and $\widehat{M}_n$ is a set-valued fuzzy mapping induced by M_n. Let Ψ be a function that satisfies Lemma 3.1.4 and for $s, t \in X$ and i, j be the positive integers, then according to the given condition:

$$H_b(\widehat{M}_i s, \widehat{M}_j y) < \Psi_b \left(\max \left(d_b(s, t), D_b(s, \widehat{M}_i s), D_b(t, \widehat{M}_j t), \frac{D_b(s, \widehat{M}_j t) + D_b(t, \widehat{M}_i s)}{2} \right) \right)$$
$$(4.16)$$

there exist a common fixed point of $\{M_n\}$.

Proof. Let s_0 be a point in X, Since $\widehat{M}_1 s_0$ is nonempty therefore there is a point $s_1 \in \widehat{M}_1 s_0$.

Let

$$\delta_b = \Psi \left(\max \left(d_b(s_0, s_1), D_b(s_0, \widehat{M}_1 s_0), D_b(s_1, \widehat{M}_2 s_1), \frac{D_b(s_0, \widehat{M}_2 s_1) + D_b(s_1, \widehat{M}_1 s_0)}{2} \right) \right)$$
$$(4.17)$$

then by above Condition (4.16) we have

$$H_b(\widehat{M}_1 s_0, \widehat{M}_2 s_1) < \delta_b \qquad (4.18)$$

Then by Lemma 4.1.3, $\exists$ an $s_2 \in X$ such that $s_2 \in \widehat{M_2}s_1$ and

$$d_b(s_1, s_2) < H_b(\widehat{M_1}s_0, \widehat{M_2}s_1)$$

$$d_b(s_1, s_2) < \delta_b$$

$$d_b(s_1, s_2) = \Psi\left(\max\left(d_b(s_0, s_1), D_b(s_0, \widehat{M_1}s_0), D_b(s_1, \widehat{M_2}s_1), \frac{D_b(s_0, \widehat{M_2}s_1) + D_b(s_1, \widehat{M_1}s_0)}{2}\right)\right)$$

$$\leq \Psi\left(\max\left(d_b(s_0, s_1), d_b(s_0, s_1), d_b(s_1, s_2), \frac{d_b(s_0, s_2) + d_b(s_1, s_1)}{2}\right)\right)$$

$$= \Psi\left(\max\left(d_b(s_0, s_1), d_b(s_1, s_2), \frac{d_b(s_0, s_2)}{2}\right)\right)$$

$$\leq \Psi\left(\max\left(d_b(s_0, s_1), d_b(s_1, s_2), b\frac{d_b(s_0, s_1 + d_b(s_1, s_2)}{2}\right)\right) \quad \text{for} \quad b \geq 1 \quad (4.19)$$

Case-i For $b = 1$

It is a case of metric space followed directly from [23].

Case-ii For $b > 1$

Suppose that,

$$\max\left(d_b(s_0, s_1), d_b(s_1, s_2), b\frac{d_b(s_0, s_1) + d_b(s_1, s_2)}{2}\right) = d_b(s_1, s_2)$$

by using the condition $\quad \psi(z) < z, \quad \forall z > 0,$

$$d_b(s_1, s_2) < \psi\left(d_b(s_1, s_2)\right) < d_b(s_1, s_2)$$

which is contradiction of our supposition.

Therefore (4.19) implies that,

$$d_b(s_1, s_2) \leq \Psi\left(\max\left(d_b(s_0, s_1), b\frac{d_b(s_0, s_1 + d_b(s_1, s_2)}{2}\right)\right) \quad (4.20)$$

Now if

$$\max\left(d_b(s_0, s_1), b\frac{d_b(s_0, s_1) + d_b(s_1, s_2)}{2}\right) = d_b(s_0, s_1)$$

by using the condition $\psi(z) < z, \quad \forall z > 0,$

$$d_b(s_1, s_2) \leq \psi d_b(s_0, s_1) < d_b(s_0, s_1) \tag{4.21}$$

if

$$\max\left(d_b(s_0, s_1), b\frac{d_b(s_0, s_1) + d_b(s_1, s_2)}{2}\right) = b\frac{d_b(s_0, s_1) + d_b(s_1, s_2)}{2}$$

by using the condition $\psi(z) < z, \quad \forall z > 0,$ (4.20) implies that

$$d_b(s_1, s_2) \leq \psi\left(b\frac{d_b(s_0, s_1) + d_b(s_1, s_2)}{2}\right)$$
$$< b\frac{d_b(s_0, s_1) + d_b(s_1, s_2)}{2}$$
$$\Rightarrow d_b(s_1, s_2) < \frac{b}{2}(d_b(s_0, s_1) + d_b(s_1, s_2))$$
$$\Rightarrow (1 - \frac{b}{2})d_b(s_1, s_2) < \frac{b}{2}d_b(s_0, s_1)$$
$$\Rightarrow (\frac{2 - b}{2})d_b(s_1, s_2) < \frac{b}{2}d_b(s_0, s_1)$$
$$\Rightarrow (2 - b)d_b(s_1, s_2) < bd_b(s_0, s_1)$$

$$\Rightarrow d_b(s_1, s_2) < \frac{b}{2 - b}d_b(s_0, s_1) \tag{4.22}$$

Therefore from 4.21 and 4.22, we have

$$d_b(s_1, s_2) < bd_b(s_0, s_1) \tag{4.23}$$

in the same manner, we make sequence $\{a_n\}, \forall n \geq 1$ such that,

$$s_n \in \widehat{M_n}s_{n-1} \tag{4.24}$$

and

$$d_b(s_{2n}, s_{2n+1}) < \Psi\left(bd_b(s_{2n-1}, s_{2n})\right), \tag{4.25}$$

$$d_b(s_{2n-1}, s_{2n}) < \Psi\left(bd_b(s_{2n-2}, s_{2n-1})\right), \tag{4.26}$$

$$\therefore d_b(s_{n+1}, s_n) < \Psi\left(bd_b(s_{n-1}, s_n)\right) \le \Psi^n(bd_b(s_0, s_1)) \quad \forall n \ge 1. \tag{4.27}$$

Thus, for $m, n(n > m)$ positive integers, we have

$$d_b(s_m, s_n) \le d_b(s_m, s_{m+1}) + \cdots + d_b(s_{n-1}, a_n)$$

$$< \Psi^m d_b(s_0, s_1) + \cdots + \Psi^{n-1}(bd_b(s_0, s_1))$$

$$= \sum_{r=m}^{n-1} \Psi^r(bd_b(s_0, s_1))$$

$$\le \sum_{r=m}^{\infty} \Psi^r(bd_b(s_0, s_1))$$

According to the following condition;

$$\sum_{n=1}^{\infty} \Psi^n d(z) < \infty \quad \text{for each} \quad z > 0,$$

$\{s_n\}$ is a Cauchy sequence.
let $s_n \to p \in X$ then by Lemma 4.1.2

$$D_b(p, \widehat{M}_n p) \le b\left(d_b(p, s_m) + D_b(s_m, \widehat{M}_n p)\right) \le b\left(d_b(p, s_m) + H_b(\widehat{M}_m s_{m-1}, \widehat{M}_n p)\right)$$

$$< b\left(d_b(p, s_m) + \Psi\left(\max\left(d_b(s_{m-1}, p), D_b(s_{m-1}, \widehat{M}_m s_{m-1}), D_b(p, \widehat{M}_n p), \frac{D_b(s_{m-1}, \widehat{M}_n p) + D_b(p, \widehat{M}_m s_{m-1})}{2}\right)\right)\right)$$

$$< b\left(d_b(p, s_m) + \Psi\left(\max\left(d_b(s_{m-1}, p), d_b(s_{m-1}, s_m), D_b(p, \widehat{M}_n p), \frac{D_b(s_{m-1}, \widehat{M}_n p) + d_b(p, s_m)}{2}\right)\right)\right)$$

$$< b\left(d_b(p, s_m) + \Psi\left(\max\left(d_b(s_{m-1}, p), d_b(s_{m-1}, s_m), D_b(p, \widehat{M}_n p), \frac{b(d_b(s_{m-1}, p) + D_b(p, \widehat{M}_n p)) + d_b(p, s_m)}{2}\right)\right)\right)$$

$\therefore$ by continuity of ψ there are following four possibilities,

$$D_b(p, \widehat{M}_n p) \le b\left(d_b(p, s_m) + d_b(s_{m-1}, p)\right), \tag{4.28}$$

$$D_b(p, \widehat{M}_n p) \le b\left(d_b(p, s_m) + d_b(s_{m-1}, s_m)\right), \tag{4.29}$$

$$D_b(p, \widehat{M_n}p) \le b\left(d_b(p, s_m) + D_b(p, \widehat{M}p)\right)$$

$$\Rightarrow (1 - b)D_b(p, \widehat{M_n}p) \le bd_b(p, s_m)$$

$$\Rightarrow D_b(p, \widehat{M_n}p) \le \frac{b}{1 - b}d_b(p, s_m) \qquad (4.30)$$

$$\text{and} \quad D_b(p, \widehat{M_n}p) \le b\left(d_b(p, s_m) + \frac{b}{2}(d_b(s_{m-1}, p) + D_b(p, \widehat{M_n}p)) + \frac{1}{2}d_b(p, s_m)\right)$$

$$(1 - \frac{b^2}{2})D_b(p, \widehat{M_n}p) \le (b + \frac{b}{2})d_b(p, s_m) + (\frac{b^2}{2})d_b(s_{m-1}, p)$$

$$\Rightarrow D_b(p, \widehat{M_n}p) \le \frac{b}{2 - b^2}(3d_b(p, s_m) + b.d_b(s_{m-1}, p)) \qquad (4.31)$$

Since $s_m \longrightarrow p$ and $\lim_{m \to \infty} s_m = p$

$\therefore$ from (4.28),(4.29),(4.30) and (4.31) $D_b(p, \widehat{M_n}p)) = 0, \Rightarrow p \in \widehat{M_n}p$

Hence by Lemma 3.1.2, M_n has a common fixed point that is p. $\qquad \square$

Remark 4.2.4. Theorem 3.2.7 becomes a special case of Theorem 4.2.3 by taking $b = 1$.

Theorem 4.2.5.

Let M and N mapped from X into $K_b(X)$ be two fuzzy mappings of a complete b-metric space (X, d_b) with a continuous $b-$metric d_b and $b \ge 1$. Let the induced mappings of M and N are $\widehat{M}, \widehat{N} : X \mapsto CB(X)$, then according to Equation (3.1),

(i) $\exists$ a sequence $\{s_n\} \in X$ such that;

$$\lim_{n \to \infty} Ms_n = \lim_{n \to \infty} Ns_n = u \quad \text{for some} \quad u \subset X$$

(ii) For $b \ge 1$,
$H_b(\widehat{M_s}, \widehat{M_t}) < \max\{H_b(\widehat{N_s}, \widehat{N_q}), \frac{b}{2}[H_b(\widehat{M_p}, \widehat{N_p}) + H_b(\widehat{M_q}, \widehat{N_q})], \frac{b}{2}[H_b(\widehat{M_t}, \widehat{N_s}) + H_b(\widehat{M_s}, \widehat{N_t})]\} \quad \forall s, t \in X, s \ne t$

(iii) $\widehat{M}X \subset \widehat{N}X$

If one of $\widehat{M}X$ and $\widehat{N}X$ is a complete subspace of X this implies that there is a unique fixed point of M and N.

Proof. Since M and N mapped from X into $K_b(X)$ be two fuzzy mappings of a complete b-metric space (X, d_b), For $b = 1$ it refers to the case of metric space proved in previous Chapter in Theorem 3.2.9.

For $b > 1$

As (i) is satisfied by $\widehat{M}X$ and $\widehat{N}X$, then $\exists$ a sequence $\{s_n\} \in X$ such that,

$$\lim_{n\to\infty} M_{s_n} = \lim_{n\to\infty} N_{s_n} = u \quad \text{for some} \quad u \subset X,$$

Let us suppose that $\widehat{N}X$ is complete, this implies that for $r \in X, \lim_{n\to\infty} \widehat{N}_{s_n} = \widehat{N}_r$ and $\lim_{n\to\infty} \widehat{M}_{s_n} = \widehat{N}_r$.

then (ii) $\Rightarrow$

$$H_b(\widehat{M}_{s_n}, \widehat{M}_r) < \max\left\{ H_b(\widehat{N}_{s_n}, \widehat{N}_r), \frac{b}{2}[H_b(\widehat{M}_{s_n}, \widehat{N}_{s_n}) + H_b(\widehat{M}_r, \widehat{N}_r)], \frac{b}{2}[H_b(\widehat{M}_r, \widehat{N}_{s_n}) + H_b(\widehat{M}_{s_n}, \widehat{N}_r)] \right\}$$

$$\text{for} \quad b > 1$$

As $n \longrightarrow \infty \Rightarrow,$

$$\Rightarrow H_b(\widehat{N}_r, \widehat{M}_r) \leq \max\left\{ H_b(\widehat{N}_r, \widehat{N}_r), \frac{b}{2}[H_b(\widehat{N}_r, \widehat{N}_r) + H_b(\widehat{M}_r, \widehat{N}_r)], \frac{b}{2}[H_b(\widehat{M}_r, \widehat{N}_r) + H_b(\widehat{N}_r, \widehat{M}_r)] \right\}$$

$$\leq \frac{b}{2} H_b(\widehat{M}_r, \widehat{N}_r)$$

$$\leq \frac{b}{2} H_b(\widehat{M}_r, \widehat{N}_r)$$

$$(1 - \frac{b}{2})H_b(\widehat{M}_r, \widehat{N}_r) \leq 0$$

$$\Rightarrow H_b(\widehat{M}_r, \widehat{N}_r) = 0$$

Hence $\widehat{M}_r = \widehat{N}_r$,

since $\widehat{M}$ and $\widehat{N}$ are weakly compatible therefore $\widehat{M}\widehat{M}_r = \widehat{M}\widehat{N}_r$ this implies that $\widehat{M}\widehat{M}_r = \widehat{N}\widehat{M}_r = \widehat{M}\widehat{N}_r = \widehat{N}\widehat{N}_r$

Finally, to prove M and N has a common fixed point first we have to prove that $\widehat{M}$ and $\widehat{N}$ has a common fixed point.

$$H_b(\widehat{M}_r, \widehat{MM}_r) \leq \max\left\{H_b(\widehat{N}_r, \widehat{NM}_r), \frac{b}{2}[H_b(\widehat{M}_r, \widehat{N}_r) + H_b(\widehat{MM}_r, \widehat{NM}_r)], \frac{b}{2}[H_b(\widehat{MM}_r, \widehat{N}_r) + H_b(\widehat{M}_r, \widehat{NM}_r)]\right\}$$

$$\leq \max\left\{H_b(\widehat{M}_r, \widehat{MM}_r), \frac{b}{2}[H_b(\widehat{M}_r, \widehat{M}_r) + H_b(\widehat{MM}_r, \widehat{MM}_r)], \frac{b}{2}[H_b(\widehat{MM}_r, \widehat{M}_r) + H_b(\widehat{M}_r, \widehat{MM}_r)]\right\}$$

$$\leq \max\left\{H_b(\widehat{M}_r, \widehat{MM}_r), bH_b(\widehat{MM}_r, \widehat{M}_r)\right\}$$

$$\leq bH_b(\widehat{M}_r, \widehat{MM}_r)$$

$$(1-b)H_b(\widehat{M}_r, \widehat{MM}_r) \leq 0$$

$$\Rightarrow H_b(\widehat{M}_r, \widehat{MM}_r) = 0$$

$$\Rightarrow \widehat{M}_r = \widehat{MM}_r \text{ also } \widehat{NM}_r = \widehat{M}_r,$$

Similarly, $\widehat{M}X$ is a complete subspace of X as $\widehat{M}X \subseteq \widehat{N}X$.

$\because r$ is a common fixed point of $\widehat{M}$ and $\widehat{N}$

$\therefore r$ is also a fixed point of M and N. $\qquad\qquad\square$

Remark 4.2.6. Theorem 3.2.9 becomes a special case of Theorem 4.2.5 by taking $b = 1$.

4.3 Conclusion

In this thesis, our work is started with the review of litrature related to fixed point theory. Several papers are reviewed but we focused on the paper [23] titled as "Common fixed point theorems for fuzzy mappings" in metric space which is the main task of our work. Then we extended it in the settings of b- metric space. First we start from already existing results then we expand them in accordance with the definition of b- metric spaces and reached at the results. These results might be helpful for solving certain problems related to fuzzy mappings in b-metric spaces.

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